Solution - PS 4

Note Title 01/04/2008

1. (7.39)

$$\frac{U}{V} = \int_0^\infty \frac{8\pi \epsilon^3 / (hc)^3}{e^{\epsilon/kT} - 1} d\epsilon$$

$$\lambda = \frac{hc}{\epsilon} \qquad d\lambda = \frac{-hc}{\epsilon^2} d\epsilon$$

$$\Rightarrow \quad d\epsilon = -\frac{\epsilon^2}{hc} d\lambda = -\left(\frac{hc}{\lambda^2}\right)^{\cancel{2}} \frac{1}{\cancel{hc}} d\lambda$$

$$\frac{U}{V} = \int_\infty^0 \left(-\frac{hc}{\lambda^2} d\lambda\right) \cdot \frac{8\pi}{\lambda^3} \frac{1}{e^{hc/\lambda kT} - 1}$$

$$= \int_0^\infty \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{e^{hc/\lambda kT} - 1} = \int_0^\infty u(\lambda) d\lambda$$

where we defined the photon spectrum

$$u(\lambda) \equiv \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}$$

To plot this, we write dimensionless variable

$$y \equiv \frac{\lambda kT}{hc}$$

Then $$\frac{U}{V} = \frac{8\pi (kT)^4}{(hc)^3} \underbrace{\int_0^\infty \frac{1/y^5}{e^{1/y} - 1} dy}_{\text{dimensionless \#}}$$

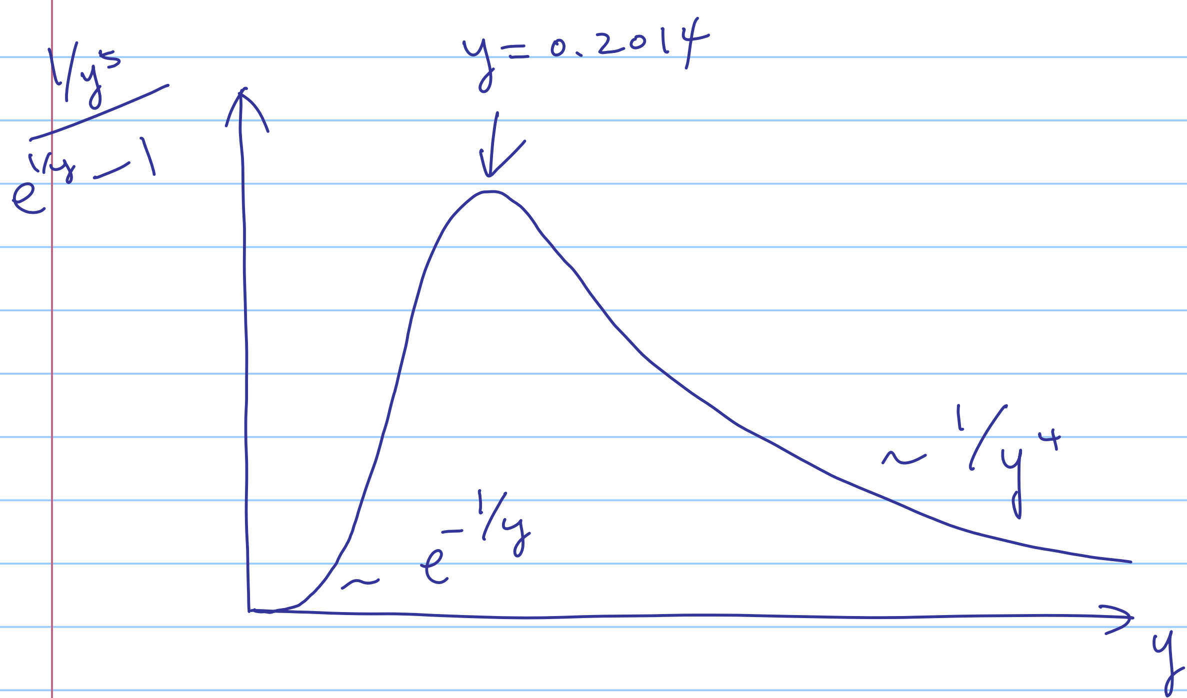
For large y, $e^{1/y} \approx 1 + \frac{1}{y}$ $\therefore u(y) \approx \frac{1}{y^4}$

small y, $e^{1/y} - 1 \approx e^{1/y}$ $\therefore u(y) \approx e^{-1/y}$

Maximum occurs at

$u'(y) = 0 \rightarrow y \approx 0.2014$

$$u(y) = \frac{1/y^5}{e^{1/y} - 1}$$



Therefore the maximum in this graph occurs at $\frac{\lambda k T}{hc} \approx 0.2014$

or $\lambda = \frac{hc}{4.97\, kT}$

which is different from $hc/2.82kT$

Why?

$u(\epsilon)$ is the energy density per unit energy, while
$u(\lambda)$ " " " per unit wavelength.
The units of photon energy and wavelength depend on each other in a nonlinear way. For instance, a one-unit photon energy range, say a range of 1eV, corresponds to a larger range of wavelengths if it's from 2eV to 3eV than it is from 200 eV to 201 eV.

Either formula, however, will give the same answer when you integrate it appropriately between any two points.

2. (7.48)

(b) We will use the Fermi-Dirac distribution with $\mu=0$:
$$\bar{n}(\epsilon) = \frac{1}{e^{\epsilon/kT}+1}$$

Note that Bose-Einstein case is jus $\left(e^{\epsilon/kT}-1\right)^{-1}$

Using the same formula for # of neutrino modes (eq. 7.80), one can obtain
$$U = \cancel{2}\cdot 3 \cdot \sum_{n_x,n_y,n_z} \frac{hcn}{\cancel{2}L} \frac{1}{e^{hcn/2LkT}+1}$$

Turning this to integral,

$$U = \frac{3hc}{L} \int_0^\infty n^2 dn \underbrace{\int_0^{\pi/2} \sin\theta \, d\theta \int_0^{\pi/2} d\varphi}_{\pi/2} \frac{n}{e^{hcn/2LkT}+1}$$

$$= \frac{3\pi hc}{2L} \int_0^\infty \frac{n^3 dn}{e^{hcn/2LkT}+1} \qquad x \equiv \frac{hcn}{2LkT}$$

$$= \frac{3\pi hc}{2L} \int_0^\infty \left(\frac{2LkT}{hc}\right)^4 \frac{x^3 dx}{e^x+1}$$

$$= 24\pi \left(\frac{L}{hc}\right)^3 (kT)^4 \underbrace{\int_0^\infty \frac{x^3 dx}{e^x+1}}_{\frac{7\pi^4}{120}} = \frac{7\pi^5 (kT)^4}{5(hc)^3}$$

(c) $N = 2 \cdot 3 \cdot \sum \frac{1}{e^{hcn/2LkT}+1}$

$$= 6 \cdot \frac{\pi}{2} \cdot \int_0^\infty \frac{n^2 dn}{e^{hcn/2LkT}+1} \Leftarrow x \equiv \frac{hcn}{2LkT}$$

$$= 3\pi \cdot \left(\frac{2LkT}{hc}\right)^3 \underbrace{\int_0^\infty \frac{x^2 dx}{e^x+1}}_{\sim 1.804} \approx 135.9 \left(\frac{LkT}{hc}\right)^3$$

Using $V = L^3$ (volume),

$$\frac{N}{V} = 135.9 \left(\frac{kT}{hc}\right)^3 = 135.9 \left(\frac{8.62\times10^{-5} \cdot (eV/K) \cdot 1.95\,K}{4.14\times10^{-15}\, eV \cdot s \cdot 3\times10^8\, m/s}\right)^3$$

$$\approx \boxed{3.4\times10^8\ m^{-3}}$$

(d) For a single species of neutrino and antineutrino, the present number density would be 1/3 of the number just calculated, or 1.1×10^8 per cubic meter. The average density of ordinary matter is about one proton per cubic meter, or, multiplying by c^2 to get the energy equivalent about 10^9 eV per cubic meter.
To equal this energy density, the neutrinos would need an energy (mc^2) of about 10 eV each, since there are roughly 10^8 of them.

(3) Visible light is roughly from 1.7eV ~ 3.1eV which is shaded in the graph. You can see that visible light is only a small fraction of the incandescent light. (Most of energy is lost through heat)