Problem Set #2 - Solutions

Note Title 17/03/2008

1. (Schroeder 2.36)

C: A Kilogram of carbon is about 83 moles, so the entropy of the book is

$$S \sim Nk \sim 83 \cdot 6\times10^{23} \times 1.38\times10^{-23} \approx 700\ \mathrm{J/K}$$

moose: A kilogram of water would be about 50 moles, so a 400-kg moose would contain about 20,000 moles of molecules. Its entropy would therefore be

$$S = (2\times10^4)\cdot(6\times10^{23})\cdot 1.38\times10^{-23} \approx 10^5\ \mathrm{J/K}$$

sun: A mole of hydrogen is only a gram, and when ionized actually contains 2 moles of ions. The sun, therefore, contains roughly 4×10^{33} moles of particles (ions), which is about 24×10^{56} particles. Note that the log term in Sakur-Tetrode equation is of order of 10, so the entropy would be

$$S = 24\times10^{56}\cdot 1.38\times10^{-23}\cdot 10 \approx 10^{36}\ \mathrm{J/K}$$

2. (Schroeder 2.42) Black hole

(a) In the SI system, the unit of G is $N\cdot m^2/kg^2$. But a newton is a $kg\cdot m/s^2$, so the unit of G can also be written as $m^3/kg\cdot s^2$.

Since M has unit of kg,
c has " m/s,

Combining these three to get "meter" unit:

$GM \rightarrow$ kills kg.
$c^2 \rightarrow$ kills s^2
$\Rightarrow \dfrac{GM}{c^2}$ has unit of m.

(b) From above, the entropy of a system is of the same order as the number of particles (multiplied by k to get the conventional entropy)

Cont.

If we take N particles and compress it to form a black hole, the second law requires that when we're done, the entropy of the black hole is still at least of order N.

But since the end result is the same whether we start with a lot of particles or a few, the final entropy must be of the order of maximum N, the largest possible number of particles that it could have been formed from.

(c) Suppose we start with N photons, with wavelength given by the size of the black hole: $\lambda = GM/c^2$

Each photon has an energy $\epsilon = hc/\lambda$, and the total energy must equal Mc^2.

$$Mc^2 = N \cdot \epsilon = N \cdot \frac{hc}{\lambda} = \frac{Nhc}{GM/c^2} = \frac{Nhc^3}{GM}$$

Solving for N: $$N = \frac{GM^2}{hc}$$

So the entropy must be of order $S \sim \frac{GM^2 k}{hc}$

(d) For a one-solar mass black hole

$$S = \frac{8\pi^2 (6.67 \times 10^{-11} \, N \cdot m^2/kg^2)(2 \times 10^{30} \, kg)^2 (1.38 \times 10^{-23} \, J/K)}{(6.63 \times 10^{-34} \, J \cdot s)(3 \times 10^8 \, m/s)}$$

$$\approx 1.5 \times 10^{54} \, J/K$$

This is an enormous entropy.

For comparison, an ordinary star like the sun contains of order 10^{57} particles, so its entropy is something like $\sim 10^{34}$ J/K. To equal the entropy of the black hole, you would need 10^{20} ordinary stars!

3. (Schroeder 3.25)

(a) Starting with the formula given for Ω

$$S = k\ln\Omega = k\ln\left(\frac{q+N}{q}\right)^q + k\ln\left(\frac{q+N}{N}\right)^N$$

$$= kq\ln\left(\frac{q+N}{q}\right) + kN\ln\left(\frac{q+N}{N}\right)$$

The omitted factors in Ω were of order $\sqrt{N}$ or $\sqrt{q}$ which are merely "large". The logarithm of such a factor is a small number, negligible compared to q or N.

(b)

$$\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{\partial S}{\partial q}\frac{\partial q}{\partial U} = \frac{1}{\epsilon}\frac{\partial S}{\partial q} \quad \leftarrow \quad U = q\epsilon$$

$$= \frac{k}{\epsilon}\frac{\partial}{\partial q}\left[q\ln(q+N) - q\ln q + N\ln(q+N) - N\ln N\right]$$

$$= \frac{k}{\epsilon}\left[\ln(q+N) + q\frac{1}{q+N} - \ln q - q\frac{1}{q} + \frac{N}{q+N}\right]$$

$$= \frac{k}{\epsilon}\left[\ln\left(\frac{q+N}{q}\right) + \cancel{\frac{q+N}{q+N}} - \cancel{\frac{q}{q}}\right] = \frac{k}{\epsilon}\ln\left(1+\frac{N}{q}\right)$$

$$T = \frac{\epsilon}{k\ln\left(1+\frac{N}{q}\right)}$$

(c) Solving for U,

$$\ln\left(1+\frac{N\epsilon}{U}\right) = \frac{\epsilon}{kT}$$

$$1+\frac{N\epsilon}{U} = e^{\epsilon/kT}$$

$$U = \frac{N\epsilon}{e^{\epsilon/kT}-1}$$

$$C = \frac{\partial U}{\partial T} = -\frac{N\epsilon}{(e^{\epsilon/kT}-1)^2}e^{\epsilon/kT}\cdot\left(-\frac{\epsilon}{kT^2}\right) = Nk\left(\frac{\epsilon}{kT}\right)^2\frac{e^{\epsilon/kT}}{(e^{\epsilon/kT}-1)^2}$$

(d) In the limit $T \to \infty$ $\frac{\epsilon}{kT} \to 0$, and $e^{\epsilon/kT} \approx 1 + \frac{\epsilon}{kT}$

Then $C \simeq Nk\cancel{\left(\frac{\epsilon}{kT}\right)^2} \cdot \frac{1}{\cancel{(\epsilon/kT)^2}} = Nk$

This is just the prediction of the equipartition theorem. (and also the result obtained in the lecture)

(e) We want to plot $\frac{C}{Nk} = \frac{e^{1/t}}{t^2(e^{1/t}-1)^2}$

with $t \equiv \frac{kT}{\epsilon}$.

For $t \to \infty$, $e^{1/t} \approx 1 + \frac{1}{t}$

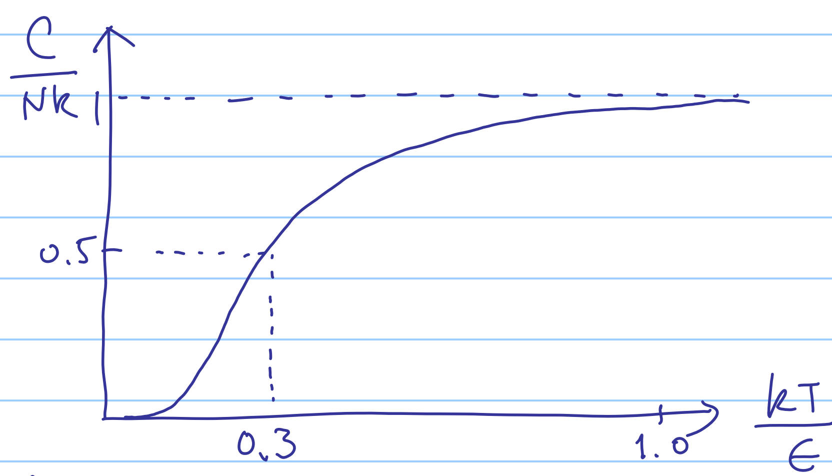
$$\frac{C}{Nk} \approx \frac{1+\frac{1}{t}}{\cancel{t^2}\left(\cancel{1}+\frac{1}{t}\cancel{-1}\right)^{2}} \approx 1$$

For $t \to 0$ the exponential term dominates,

so $\frac{C}{Nk} \approx \frac{e^{1/t}}{e^{2/t}} \approx e^{-1/t}$

which is exponentially suppressed.

Therefore



You can also obtain this with your favorite computer program too.

Comparing the above curve to the data in Fig 1.14, we see that it is generally similar but not exactly the same. We can in fact neglect the discrepancy at high temperatures where the measured heat capacity gradually rise above the equipartition prediction, because this is a feature of C_p, not C_V.

At low temperatures, however, there is a subtle but significant difference: The curve predicted by the Einstein model is much flatter as it approaches $T \to 0$. (Note the exponential suppression $e^{-1/t}$) This is the result of the simplification made in the Einstein model. Later in the course, you will learn that in realistic model (Debye), $C \sim T^3$ as $T \to 0$.

Anyhow, we can estimate the value of ϵ by noting that the heat capacity reaches half its equipartition value at $kT \approx \epsilon/3$. For lead, this temperature is about 22K, so $\epsilon \approx 3 \cdot (8.6 \times 10^{-5}\ \text{eV/K})(22\text{K}) = \underline{0.0057\ \text{eV}}$. For aluminum it's about 100K, so $\underline{\epsilon \approx 0.026\text{eV}}$. And for diamond, it's about 460K, so $\epsilon = 0.12\text{eV}$. Because ϵ is proportional to the frequency of each atomic oscillator, and stiff/light materials vibrate at higher frequencies, it makes sense that ϵ would be lowest for lead and highest for diamond.

4. (Einstein solid)

Note: Here we will consider that 20 atoms mean that there are 20 oscillators. In 3D, one can associate 3 oscillators per atom. But we will not worry about that here.

(a) The entropy of an Einstein solid with $N=20$ and $q=11, 10, 9$ is

$$\frac{S(11\varepsilon)}{k} = \ln \Omega(N=20, q=11) = \ln \frac{(20+11-1)!}{11!\,19!} \approx 17.8$$

$$\frac{S(10\varepsilon)}{k} = \ln \Omega(N=20, q=10) = \ln \frac{(20+10-1)!}{10!\,19!} \approx 16.8$$

$$\frac{S(9\varepsilon)}{k} = \ln \frac{(20+9-1)!}{9!\,19!} \approx 15.7$$

Then using the 1st method, we obtain

$$T = \left(\frac{\partial S}{\partial U}\right)^{-1} = \frac{\varepsilon}{S(11\epsilon)-S(10\epsilon)} = \frac{0.02\,(\mathrm{eV})}{k(17.8-16.8)}$$

$$= \frac{0.02\,(\mathrm{eV})}{8.62\times10^{-5}\,(\mathrm{eV}/\mathrm{K})} \simeq 231\,(\mathrm{K})$$

Using the 2nd method,

$$T = \left(\frac{\partial S}{\partial U}\right)^{-1} = \frac{2\varepsilon}{S(11\epsilon)-S(9\epsilon)} = \frac{0.04\,(\mathrm{eV})}{k(17.8-15.7)}$$

$$= \frac{0.04\,(\mathrm{eV})}{2.1\cdot 8.62\times10^{-5}\,(\mathrm{eV}/\mathrm{K})} \approx 224\,(\mathrm{K})$$

The results are similar!

(b) Using $U = NkT$, we obtain

$$T = \frac{U}{Nk} = \frac{10\epsilon}{Nk} = \frac{0.2\ (eV)}{20 \times 8.62 \times 10^{-5}\ (eV/K)} \approx \underline{116 (K)}$$

This is very different from the part (a) result. This is not surprising given that $U = NkT$ works only for large $q \gg N \gg 1$

(c) For $N = 200$, $q = 100$,

$$\frac{S(101\epsilon)}{k} = \ln \frac{(200+101-1)!}{101!\ 199!} \approx 188.6$$

$$\frac{S(100\epsilon)}{k} = \ln \frac{(200+100-1)!}{100!\ 199!} \approx 187.5$$

$$T = \left(\frac{\partial S}{\partial U}\right)^{-1} = \frac{\epsilon}{S(101\epsilon) - S(100\epsilon)} = \frac{0.02\,(eV)}{k\,(188.6 - 187.5)}$$

$$\approx \frac{0.02}{1.1 \times 8.62 \times 10^{-5}} \approx \underline{\underline{213\ (K)}}$$

$$T = \frac{U}{Nk} = \frac{100\,\epsilon}{Nk} = \frac{2\,(eV)}{200 \times 8.62 \times 10^{-5}} \approx \underline{\underline{116\ (K)}}$$

These results are still very different, since we are not in the limit of $q \gg N \gg 1$.