Note Title 10/03/2008

Problem Set #1 — Solutions

(1) (1.11 Schroeder)

From the ideal gas eq., the volume per molecule is

$$\frac{V}{N} = \frac{kT}{P} = \frac{(1.38\times10^{-23}\ J/K)(300\,K)}{10^5\ N/m^2}$$

$$\cong 4.1\times10^{-26}\ m^3 = \boxed{41\ nm^3}$$

N.B. We used room temperature: 300 K
atmospheric pressure: $\sim 10^5\ N/m^2$

If we imagine each molecule being a cube of this volume, then the width of the cube would be $\sqrt[3]{41} \sim 3.5$ nm; this is a good estimate of the average distance. The diameter of a molecule like N_2 or H_2O is only a few angstroms. (Mostly the bond length between the atoms) Therefore, the size of a molecule is much smaller than the distance between molecules.

(2) (1.23 Schroeder)

Helium has three degrees of freedom (translational).

So $U = 3N\cdot\frac{1}{2}kT = \frac{3}{2}PV$ (ideal gas law)

For $P = 10^5\ N/m^2$ (atmospheric pressure) and $V = 10^{-3}\ m^3$,

$$\boxed{U = 150\,J}$$

For air (nitrogen + oxygen), there are 5 degrees of freedom at room temperature.

Then, thermal energy is $U = \frac{5}{2}PV = \boxed{250\,J}$

③ (2.3 Schroeder)

(a) There are two possible states for the 1st coin, two for the second, ... So the total number of micro states is $2^{50} \approx \boxed{1.13\times10^{15}}$

(b) $\Omega(25) = \binom{50}{25} = \dfrac{50!}{25!\,25!} = \boxed{1.26\times10^{14}}$

(c) The probability is $\dfrac{\Omega(25)}{\Omega(\text{all})} = \dfrac{1.26\times10^{14}}{1.13\times10^{15}} \approx \boxed{0.11}$

(d) $P(30) = \dfrac{\Omega(30)}{\Omega(\text{all})} = \dfrac{1}{2^{50}}\cdot\dfrac{50!}{30!\,20!} \approx \boxed{0.042}$

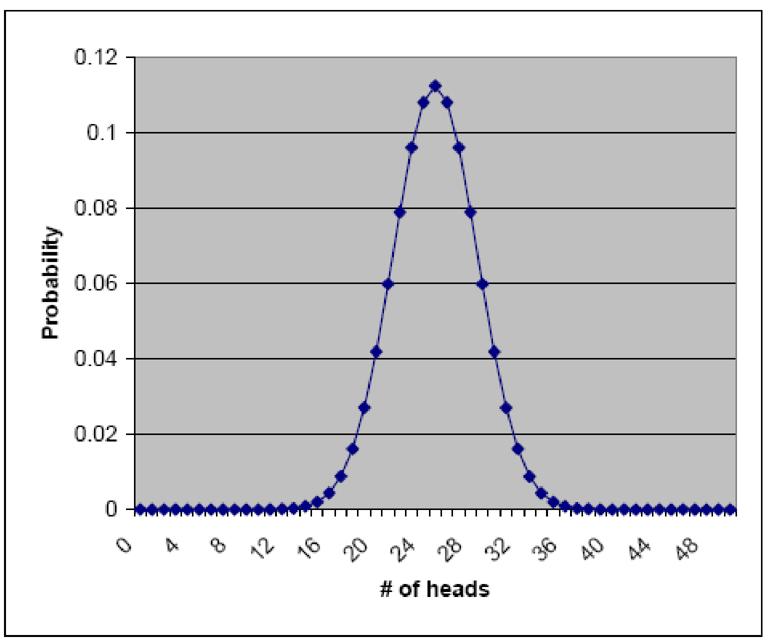
(e) $P(40) = \dfrac{1}{2^{50}}\,\dfrac{50!}{40!\,10!} = \boxed{9.1\times10^{-6}}$

(f) $P(50) = \dfrac{1}{2^{50}} \approx \boxed{8.88\times10^{-16}}$

(g)

# of heads	Multiplicity	Probability
0	1	8.88178E-16
1	50	4.44089E-14
2	1225	1.08802E-12
3	19600	1.74083E-11
4	230300	2.04547E-10
5	2118760	1.88184E-09
6	15890700	1.41138E-08
7	99884400	8.87152E-08
8	536878650	4.76844E-07
9	2505433700	2.22527E-06
10	1.0272E+10	9.12362E-06
11	3.7354E+10	3.31768E-05
12	1.214E+11	0.000107825
13	3.5486E+11	0.000315179
14	9.3785E+11	0.000832974
15	2.2508E+12	0.001999138
16	4.9237E+12	0.004373115
17	9.8474E+12	0.00874623
18	1.8054E+13	0.016034755
19	3.0406E+13	0.027005903
20	4.7129E+13	0.041859149
21	6.7327E+13	0.059798785
22	8.875E+13	0.078825671
23	1.0804E+14	0.095961686
24	1.2155E+14	0.107956897

⋮



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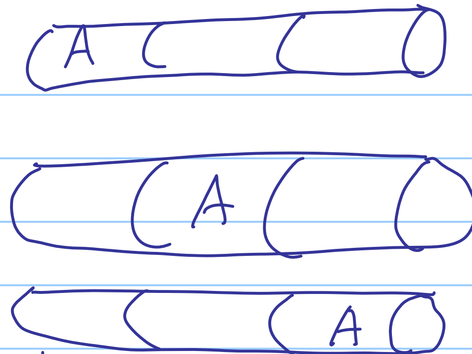
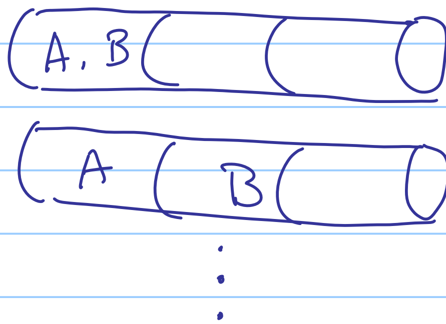
(a) 3 microstates :



(b) First, A can be in one of the three regions, and then B can also occupy 3 regions, Therefore $3 \times 3 = 9$ microstates



(c) A has 3 ways to occupy the regions, and B, and C as well. So $3 \times 3 \times 3 = 27$ microstates

(d) 3^N

(e) For one molecule : 4

For two molecules : $4 \times 4 = 16$

For three molecules : $4 \times 4 \times 4 = 64$

For N molecules : 4^N

(f) The number of microstates (multiplicity) of N gas molecules should be proportional to V^N. This result will be used in our discussion of ideal gas.

⑤ (2.24 Schroeder)

(a) The mostly likely macrostate is $N_\uparrow = N/2$.
The multiplicity is

$$\Omega_{max} = \frac{N!}{N_\uparrow! N_\downarrow!} = \frac{N!}{(N/2)!} \approx \frac{N^N e^{-N}\sqrt{2\pi N}}{\left[\left(\frac{N}{2}\right)^{N/2} e^{-N/2}\sqrt{\pi N}\right]^2}$$

$$= \frac{\cancel{N^N} \cancel{e^{-N}} \sqrt{2\pi N}}{\left(\frac{\cancel{N}}{2}\right)^N \cdot \cancel{e^{-N}} (\pi N)} = 2^N \cdot \sqrt{\frac{2}{\pi N}}$$

Note that $\sqrt{\frac{2}{\pi N}}$ is small compared to 2^N.

(b) By Stirling's approximation, the multiplicity is

$$\Omega = \frac{N!}{N_\uparrow! N_\downarrow!} \approx \frac{N^N \cancel{e^{-N}} \sqrt{2\pi N}}{N_\uparrow^{N_\uparrow} \cancel{e^{-N_\uparrow}} \sqrt{2\pi N_\uparrow} \cdot N_\downarrow^{N_\downarrow} \cancel{e^{-N_\downarrow}} \sqrt{2\pi N_\downarrow}}$$

$$= \frac{N^N}{N_\uparrow^{N_\uparrow} N_\downarrow^{N_\downarrow}} \sqrt{\frac{N}{2\pi N_\uparrow N_\downarrow}}$$

Setting $N_\uparrow = \frac{N}{2} + x$ and $N_\downarrow = \frac{N}{2} - x$, we obtain

$$\Omega \cong \frac{N^N}{\left(\frac{N}{2}+x\right)^{\frac{N}{2}+x}\left(\frac{N}{2}-x\right)^{\frac{N}{2}-x}} \sqrt{\frac{N}{2\pi\left(\frac{N}{2}+x\right)\left(\frac{N}{2}-x\right)}}$$

$$= \frac{N^N}{\left(\frac{N^2}{4}-x^2\right)^{N/2}\left(\frac{N}{2}+x\right)^{x}\left(\frac{N}{2}-x\right)^{-x}} \sqrt{\frac{N}{2\pi\left(\frac{N^2}{4}-x^2\right)}}$$

At this point, it becomes simpler if we work with the logarithm of Ω:

$$\ln\Omega = N\ln N - \frac{N}{2}\underbrace{\ln\left[\frac{N^2}{4}-x^2\right]}_{1} - x\underbrace{\ln\left(\frac{N}{2}+x\right)}_{2} + x\underbrace{\ln\left(\frac{N}{2}-x\right)}_{2}$$

$$+\frac{1}{2}\ln\left(\frac{N}{2\pi}\right) - \frac{1}{2}\ln\left[\frac{N^2}{4}-x^2\right]$$

Now, using Taylor series for $x \ll N$

1 $$\ln\left[\frac{N^2}{4}-x^2\right] = \ln\frac{N^2}{4} + \ln\left(1-\left(\frac{2x}{N}\right)^2\right)$$

$$\approx 2\ln\frac{N}{2} - \left(\frac{2x}{N}\right)^2$$

2 $$\ln\left(\frac{N}{2}\pm x\right) \simeq \ln\frac{N}{2} + \ln\left(1 \pm \frac{2x}{N}\right)$$

$$\approx \ln\frac{N}{2} \pm \frac{2x}{N}$$

Then,

$$\ln\Omega = N\ln N - N\ln\frac{N}{2} + \cancel{\frac{N}{2}\cdot\frac{4x^2}{N^2}} - \cancel{x\cdot\frac{2x}{N}} + x\left(-\frac{2x}{N}\right)$$

$$+\frac{1}{2}\ln\left(\frac{N}{2\pi}\right) - \ln\frac{N}{2} + \frac{2x^2}{N^2}$$

$$= N\ln 2 - \frac{2x^2}{N} + \frac{1}{2}\ln\left(\frac{2}{\pi N}\right) - \frac{2x^2}{N^2}$$

Ignoring the last term, we obtain for $x \ll N$

$$\Omega = 2^N \cdot e^{-2x^2/N} \cdot \sqrt{\frac{2}{\pi N}}$$

This is a Gaussian fn. The peak is at $x=0$, and the value at $x=0$ agrees with the result of part (a).

* Note that if we ignore the $\sqrt{2\pi N}$ part of the Stirling's approximation, we will get 2^N for part (a), and part (b) will become

$$\Omega = 2^N e^{-2x^2/N}$$

These are acceptable answers.

(c) The Gaussian fn. falls off to $1/e$ of its peak value when $2x^2/N = 1$, or $x = \sqrt{\frac{N}{2}}$

So the full width of the peak would be $\boxed{\sqrt{2N}}$

Again $\sqrt{N}$ factor!

(d) For $N = 10^6$, the half width of the peak would be $\sqrt{500{,}000} \approx 700$. So 1000 would be just a little beyond the point where the Gaussian falls off to $1/e$ of its maximum. I would not be surprised to obtain 1000 heads.

On the other hand, 10,000 lies far outside of the peak. At this point, the Gaussian has fallen off to $e^{-200} \approx 10^{-87}$ of its maximum value. I would be very surprised if I get this (10,000 heads) result.