Solution - Midterm (March 28th)

Note Title 25/03/2008

1\. (a) $S = k \ln \Omega$

$$= k \ln \left[\frac{1}{N!} \frac{A^N \pi^N}{h^{2N} N!} (2m)^N U^N \right]$$

$$= \underbrace{Nk \ln A}_{} + \underbrace{Nk \ln U}_{} + k \ln \left[\left(\frac{2m\pi}{h^2} \right)^N \frac{1}{(N!)^2} \right]$$

(b) $\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{Nk}{U}$

$\therefore \; U = NkT$

(c) $C_v = \frac{dU}{dT} = Nk$:

equipartition theorem: for 2D gas with only translational deg. of freedom.
$C_v = Nk$ (2 deg of freedom)
$\therefore$ The result is in agreement with the equipartition theorem.

2\. (a) $E = 0$: doubly degenerate

$$\therefore Z = e^{-\mu B/kT} + e^{\mu B/kT} + 2$$

$$= 2\cosh\left(\frac{\mu B}{kT}\right) + 2 = 2\left(1 + \cosh\left(\frac{\mu B}{kT}\right)\right)$$

(b) Probability = $\frac{\text{Boltzmann factor}}{\text{part. fn.}} = \frac{e^{\mu B/kT}}{Z}$

(c) $F = -kT\ln Z = -kT\ln\left\{2\left[1+\cosh\left(\frac{\mu B}{kT}\right)\right]\right\}$

$= -kT\ln 2 - kT\ln\left[1+\cosh\left(\frac{\mu B}{kT}\right)\right]$

(d) $\bar{m} = P(\uparrow)\,m(\uparrow) + \cdots\cdots$

$= \frac{e^{\mu B/kT}}{Z}\cdot(\mu) - \mu\frac{e^{-\mu B/kT}}{Z} + 0$

$= \frac{2\mu\sinh\left(\frac{\mu B}{kT}\right)}{2\cosh\left(\frac{\mu B}{kT}\right)+2} = \frac{\mu\sinh\left(\frac{\mu B}{kT}\right)}{1+\cosh\left(\frac{\mu B}{kT}\right)}$

Alternatively, one can also obtain this by

$\bar{m} = -\left(\frac{\partial F}{\partial B}\right)_{N,T} = +kT\frac{\partial}{\partial B}\ln\left[1+\cosh\left(\frac{\mu B}{kT}\right)\right]$

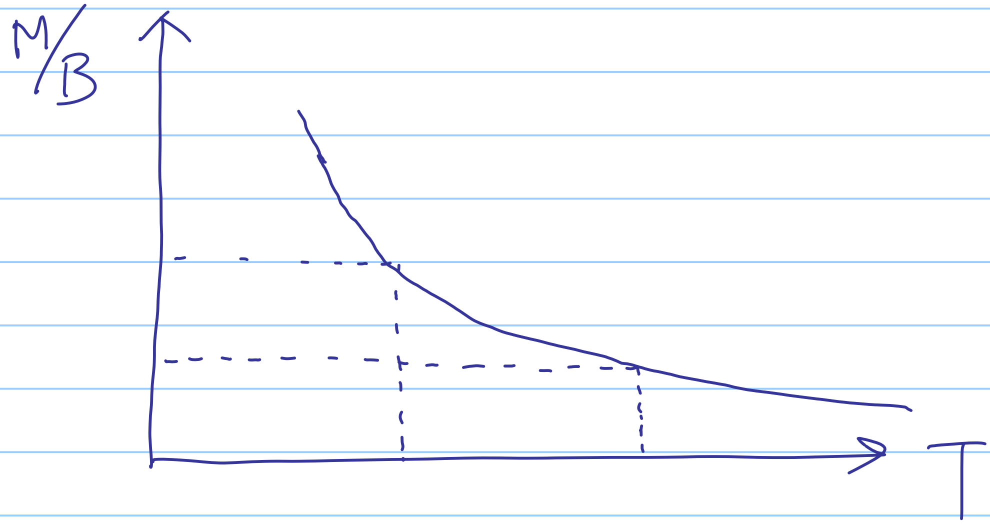
$= \cancel{kT}\cdot\frac{\sinh\left(\frac{\mu B}{kT}\right)\cdot\frac{\mu}{\cancel{kT}}}{1+\cosh\left(\frac{\mu B}{kT}\right)} = \frac{\mu\sinh\left(\frac{\mu B}{kT}\right)}{1+\cosh\left(\frac{\mu B}{kT}\right)}$

(e) $M = N\bar{m} = \frac{N\mu\sinh\left(\frac{\mu B}{kT}\right)}{1+\cosh\left(\frac{\mu B}{kT}\right)}$ $\quad T\to\infty$

$M \approx \frac{N\mu\cdot\left(\frac{\mu B}{kT}\right)}{1+1} = \frac{N\mu^2 B}{2kT}$

$\therefore \frac{M}{B} \approx \frac{N\mu^2}{2kT}$

The M/B vs. T graph is



(f) In the highest temperature limit, the most probable state is that of total random spin orientation. That is, the maximum multiplicity of 4-state paramagnet is 4^N. Therefore, the entropy in this case is

$$S = k \ln 4^N = \underline{Nk \ln 4}$$

Alternatively, one can obtain S from F: Since $F = -NkT \ln 2 - NkT \ln(1+1)$ in the high T limit, (Note the factor N to account for N spins)

$$S = -\left(\frac{\partial F}{\partial T}\right)_{B,N} = \frac{\partial}{\partial T}\left\{ NkT \ln 2 + NkT \ln 2 \right\}$$

$$= \underline{Nk \ln 4}$$

3. From $C_v = \frac{dU}{dT}$ and $\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_{V,N}$

$$dS = \frac{C_v}{T} dT$$

$$\Delta S = \int_{T_i}^{T_f} \frac{C_v}{T} dT = \int_0^{T_f} (a + bT^2)\, dT = aT + \frac{b}{3}T^3 \Big|_0^{T_f}$$

$\leftarrow T_i = 0$

$$S(T) - \cancel{S(0)} = aT + \frac{b}{3}T^3$$

$$S(10K) = 0.00135 \cdot 10 + \frac{2.48 \times 10^{-5}}{3} \cdot 10^3$$

$$= 0.0135 + 0.827 \times 10^{-2}$$

$$= 2.18 \times 10^{-2} \ (J/K)$$

4. For random walk, the width of the distribution (or standard deviation) is roughly given by $\sqrt{N}$, where N is the number of steps.

To have 36% or more chance to reach 300 steps, the standard deviation must be close to 300, which requires that the drunk must take 300^2 steps, which is about 25 hours. (~1 day)

An answer between 12–48 hours will be accepted.