Solutions - Midterm Phy292 (2007)

① $\text{Prob} = \dfrac{\text{Boltzmann factor}}{\text{Partition fn}} = \dfrac{0.5}{2} = \dfrac{1}{4}$

② (a) $S = k \ln \Omega$

$$= k \ln \left(\frac{eN}{q}\right)^q = \boxed{kq \ln\left(\frac{eN}{q}\right)} \boxed{= \frac{kU}{\varepsilon} \ln\left(\frac{eN\varepsilon}{U}\right)}$$

(b) $$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_N = \frac{\partial}{\partial U}\left[\frac{kU}{\varepsilon}\ln\left(\frac{eN\varepsilon}{U}\right)\right]$$

$$= \frac{k}{\varepsilon}\ln\left(\frac{eN\varepsilon}{U}\right) + \frac{k\cancel{U}}{\varepsilon}\,\frac{\cancel{U}}{\cancel{eN\varepsilon}}\left(-\frac{\cancel{eN\varepsilon}}{\cancel{U^2}}\right)$$

$$= \frac{k}{\varepsilon}\left(\cancel{\ln e} + \ln\left(\frac{N\varepsilon}{U}\right) - \cancel{1}\right) = \frac{k}{\varepsilon}\ln\left(\frac{N\varepsilon}{U}\right)$$

$$\frac{\varepsilon}{kT} = \ln\left(\frac{N\varepsilon}{U}\right)$$

$$e^{\varepsilon/kT} = \frac{N\varepsilon}{U}$$

$$\therefore \quad U = N\varepsilon\, e^{-\varepsilon/kT}$$

(c) $$C_v = \left(\frac{\partial U}{\partial T}\right)_N = \frac{\partial}{\partial T}\left(N\varepsilon e^{-\varepsilon/kT}\right)$$

$$= N\varepsilon e^{-\varepsilon/kT}\left(\frac{+\varepsilon}{kT^2}\right) = Nk\left(\frac{\varepsilon}{kT}\right)^2 e^{-\varepsilon/kT}$$

(d) When $T \to 0$, $e^{-\varepsilon/kT} \to 0$,

Therefore $C_v \to 0$

(e) $$C_v = \underbrace{2N}_{\#\text{ of degrees of freedom}} \cdot \frac{1}{2}k = Nk$$

(f) They are different, because the vibrational degrees of freedom freeze out at this low temperature, and the equipartition theorem does not work.

3. (a) $\Omega_{real} \propto A^N$

(b) Ω will depend on the momentum space volume. In 2D, the momentum space is the surface area of hypersphere in 2N dimension with radius $\sqrt{2mU}$

Therefore, $\Omega \propto \left(\sqrt{2mU}\right)^{2N-1}$

Since $N \gg 1$,

$$\boxed{\Omega \propto U^N}$$

(c) $\Omega_{total} \propto \Omega_{real}\,\Omega_{momentum} \propto A^N U^N$

$\therefore$ $\Omega_{total} = f(N) A^N U^N$

(d) $S = k \ln \Omega = k \ln [f(N) A^N U^N]$

$= Nk \ln A + Nk \ln U + k \ln f(N)$

4. (a) $\Omega(N, r, l) = \dfrac{N!}{r!\, l!}$ (This is binomial distribution)

(b) To get $x=8$, r has to be 8, $l=0$.

$\therefore$ $\Omega = \dfrac{8!}{8!\,0!} = 1$

(c) For $x=0$, $r=4$, $l=4$, $\Omega = \dfrac{8!}{4!\,4!} = \dfrac{\cancel{8}^{2}\cdot 7\cdot \cancel{6}\cdot 5}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = \boxed{70}$