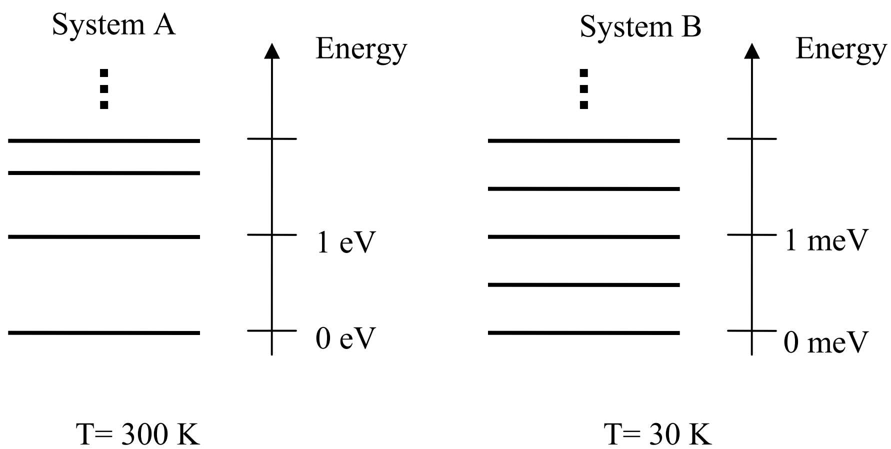
PHY 292S **Final test** **Winter 2007**

Friday, February 16th, 8:15am - 9:55am

PART I: Provide short answers to the following questions. To obtain full marks, you should provide both correct reasoning and answers. Partial marks will be considered only for numerical mistakes.



1. Consider the two (fictitious) systems with energy levels shown in the above figure. System A is in thermal equilibrium at T=300 K, while system B is in thermal equilibrium at T=30 K. We also assume that these systems are described by Boltzmann distribution function, and the energy levels are not degenerate (Note 1 meV=10^{-3} eV). The given information is not enough for calculating partition functions explicitly. However, one can easily figure out which system has a larger partition function by comparing the energy scale and the temperature. Which system's partition function is larger? Provide your explanation. (5 points)

2. In the early universe, the temperature was so high that the proton and the neutron can be thought of as two different states of the same particle, called the "nucleon". Since the neutron's rest mass is higher than the proton's by 2.3×10^{-30} kg, its energy is higher by this amount times c^2. Suppose, then, that at some very early time, the nucleons were in thermal equilibrium with the rest of universe at 1.5×10^{11} K. What was the ratio of the number of neutrons to the number of protons? (5 points)

3. The solar radiation received by Mars is 600 W/m^2. Unlike the earth, Mars does not have much of an atmosphere to reflect the radiation back to the space. If we assume that Mars acts as a perfect blackbody, what is its surface temperature? (5 points)

4. Consider an ideal gas system made of $N=10^{23}$ helium atoms in a volume of $V=1\ cm^3$. If this system is in thermal equilibrium at temperature T, what is the condition for the classical (Boltzmann) statistics to be valid? In other words, find the temperature range for which you do not need to use quantum statistics. (Hint: Write your answer as $T \gg ?$ or $T \ll ?$) (5 points)

5. The following table has information about various physical properties of three materials which have the same crystal structure: diamond (C), Si, and Ge. Based on the information given, which material has the largest heat capacity at room temperature? Provide the reason. (5 points)

	Lattice parameter (Å)	Density (g/cm^3)	Debye temperature (K)	Band gap (eV)
Diamond	3.567	3.516	1860	5.4
Si	5.430	2.33	625	1.17
Ge	5.658	5.32	360	0.744

PART II: The following questions require mathematical derivations. Provide all the mathematical steps leading to your answer.

6. Consider the rotational motion of a diatomic molecule. Rotational energies are quantized, and the allowed rotational energies are given by $E(J)=J(J+1)\varepsilon$, where J can be 0, 1, 2,…, and ε is a constant energy unit related to the rotational inertia. The number of degenerate states for level J is (2J+1). Remember that this is the problem discussed in the lecture and in the problem set. For this problem, we will specifically consider a *single* molecule made of *distinguishable* atoms, such as NO, HCl, etc.

   (a) Write the partition function as a sum over J. (4 points)

   (b) In the low temperature limit ($kT \ll \varepsilon$), each term in the part (a) is much smaller than the one before. Since the first term is independent of T, we will cut off the sum after the second term. What is the partition function in this low temperature limit? (4 points)

   (c) Calculate the average energy in this low temperature limit. Keep only the leading order term in your answer for the next part. (4 points)

   (d) Calculate the heat capacity in this low temperature limit using the result obtained in part (c). (4 points)

   (continued on the next page)

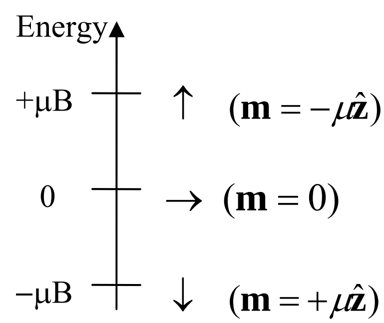
(e) In the high temperature limit (kT >> ε), we can approximate the sum in part (a) as an integral. Evaluate the integration to obtain the partition function in this high temperature limit. (4 points)

(f) Calculate the average energy in this high temperature limit. (3 points)

(g) Calculate the heat capacity in this high temperature limit. (3 points)

(h) Sketch the temperature dependence of heat capacity of diatomic molecules in the low and high temperature limit. (4 points)

7. A paramagnet consists of N noninteracting magnetic moments $\mathbf{m}$, which are located at the fixed positions of the N atoms in a crystal lattice. Each magnetic moment has, in an external magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$, three allowed states with energies $-\mu B$, 0, and $+\mu B$, as shown in the figure. In other words, we are considering a three-state paramagnet, where the angle between the magnetic moment and the magnetic field is allowed by quantum mechanics to be 0°, 90°, and 180° from the vertical, for which magnetic moments are $\mathbf{m} = +\mu\hat{\mathbf{z}}$, 0, and $-\mu\hat{\mathbf{z}}$, respectively. [There is no degeneracy.] The total magnetization $\mathbf{M}$ is of course given by $\mathbf{M} = \sum \mathbf{m}$, and the total energy is $U = -MB$.



(a) Calculate the partition function of the three-state paramagnet at temperature T. (5 points)

(b) Calculate the free energy F. (5 points)

(c) Use the Boltzmann probability distribution to calculate the average value, $\overline{M}$, of the magnetization. [Since the magnetic moments are non-interacting, to simplify things, consider the average magnetization for a single magnetic moment first.] (5 points)

(d) Sketch the qualitative behaviour of $\overline{M}$ as a function of $\mu B/kT$. Compare to the Curie law in the appropriate temperature range. (5 points)

8. In this problem, we will consider a transmission line of length L on which electromagnetic waves satisfy the one-dimensional (1D) wave equation $c^2 \partial^2 E / \partial x^2 = \partial^2 E / \partial t^2$, where E is an electric field component. This is equivalent to one-dimensional box containing photons in thermal equilibrium at temperature T. In this 1D box, the number of modes can be found by considering a standing wave solution, the energy of which is given by $\varepsilon = hf = \frac{hc}{2L} n$, with n=0,1,2,… Here, *f, h, c* are the frequency of the wave, the Planck's constant, and the speed of light, respectively.

   (a) Using the Planck's distribution function given in the formulae sheet, calculate the total energy as a function of T. [Hint: Turn the summation into integration, and use the integrals given in the formulae sheet. You can ignore the polarization factor for this question.] (10 points)

   (b) Calculate the heat capacity as a function of temperature. (5 points)

   (c) What is the entropy of the 1D photon gas at temperature T? (5 points)

   (d) In the big bang theory, the photon gas has cooled down from the early universe through adiabatic (constant-entropy) expansion. Hypothetically, let's imagine a big bang in one-dimensional universe, and assume that this photon gas in 1D was initially in thermal equilibrium at 3000 K and the length of the 1D universe was 1 m. If the temperature of the photon gas has cooled down to 3 K after adiabatic expansion, what would be the current length of the 1D universe? (5 points)