Problem Set 7
Part 2: Assignment 2

PHY 291 H1S

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Problem 9.6 *Probability current in the "sloshing" state of the infinite well.*

We work with the state given by Eq. 8-5, a balanced superposition of the two lowest energy eigenstates for the infinite well with boundaries at $x = \{0, L\}$:

$$\Psi(x,t) = \frac{1}{\sqrt{L}}\left(\sin\left(\frac{\pi x}{L}\right)e^{-i\omega_1 t} + \sin\left(\frac{2\pi x}{L}\right)e^{-i\omega_2 t}\right)$$

(a) The probability current $J(x,t)$ of the state $\Psi(x,t)$ is defined by

$$J(x,t) = \frac{-i\hbar}{2m}[\Psi^*(x,t)\partial_x\Psi(x,t) - \Psi(x,t)\partial_x\Psi^*(x,t)].$$

To evaluate this quantity, we look at the first term:

$$\begin{aligned}\Psi^*\partial_x\Psi = \frac{\pi}{L^2}\quad\Big[\quad & \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right) \quad + \quad \cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)e^{+i(\omega_2-\omega_1)t} \\ & + 2\sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi x}{L}\right) + 2\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{2\pi x}{L}\right)e^{-i(\omega_2-\omega_1)t}\quad\Big].\end{aligned}$$

Also, $\Psi\partial_x\Psi^* = (\Psi^*\partial_x\Psi)^*$. Thus, when subtracting the two in the expression for the probability current, the leftmost terms cancel and we are left with

$$\begin{aligned}J(x,t) = \frac{\pi\hbar}{mL^2}&\left[\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right) - 2\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{2\pi x}{L}\right)\right] \\ &\times\frac{1}{2i}\left(e^{+i(\omega_2-\omega_1)t} - e^{-i(\omega_2-\omega_1)t}\right),\end{aligned}$$

which simplifies to

$$J(x,t) = \frac{\pi\hbar}{mL^2}\left[\frac{3}{2}\sin\left(\frac{\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{3\pi x}{L}\right)\right]\sin\left((\omega_2-\omega_1)t\right).$$

Substituting $x = \frac{L}{2}$, the current at the center of the well is

$$J(\tfrac{L}{2},t) = \frac{2\pi\hbar}{mL^2}\sin\left((\omega_2-\omega_1)t\right).$$

(b) From problem 8-1(d), the time-dependent probability for finding the particle in the left half of the well is

$$P(0 \le x \le \tfrac{L}{2}) = \frac{1}{2} + \frac{4}{3\pi}\cos\left((\omega_2 - \omega_1)t\right).$$

The rate at which probability flows *out* of the left half is

$$\begin{aligned} -\partial_t P &= \frac{4}{3\pi}(\omega_2 - \omega_1)\sin\left((\omega_2 - \omega_1)t\right) \\ &= \frac{2\pi\hbar}{mL^2}\sin\left((\omega_2 - \omega_1)t\right) \end{aligned}$$

which is indeed equal to the current at the center of the well, $J(\frac{L}{2}, t)$.

Problem 9.8 *Conditions for a "zero" in probability current.*

The probability current is given by

$$J(x,t) = \frac{-i\hbar}{2m}\left[\Psi^*(x,t)\partial_x\Psi(x,t) - \Psi(x,t)\partial_x\Psi^*(x,t)\right] \quad .$$

(a) Sufficent.
Given, at a particular position x_0 and time t_0, $\Psi(x_0, t_0) = 0$, $\Psi^*(x_0, t_0) = 0$. Thus, $J(x_0, t_0) = 0$ as expected.

(b) Insufficient.
Assume a plane wave $\Psi(x,t) = Ae^{ikx - i\omega t}$, it satifies the condition $|\Psi(x_0, t_0)|^2 = |\Psi(x_0, t)|^2$ for any t. The spatial derivative of the plane wave is

$$\partial_x\Psi(x,t) = ikAe^{ikx - i\omega t} \quad ,$$

then the probability current is then

$$\begin{aligned} J(x,t) &= \frac{-i\hbar}{2m}[(A^* e^{-ikx + i\omega t})(ikAe^{ikx - i\omega t}) - (Ae^{ikx - i\omega t})(-ikA^* e^{-ikx + i\omega t})] \\ &= \frac{\hbar k}{m}|A|^2 \neq 0 \end{aligned}$$

at position x_0 and time t_0.

(c) Insufficient.
Using same argument as in (b).

(d) Sufficient.
The wavefunction satisfying $\Psi(x_0 + b, t_0) = \Psi(x_0 - b, t_0)$ is symmetric under mirror transformation about x_0. Looking at the spatial derivative of the wavefunction at that point:

$$\begin{aligned}
\partial_x \Psi(x, t_0)|_{x=x_0} &= -\partial_b \Psi(x_0 - b, t_0)|_{b=0} && \text{(using chain rule)} \\
&= -\partial_b \Psi(x_0 + b, t_0)|_{b=0} && \text{(symmetric condition)} \\
&= -\partial_x \Psi(x, t_0)|_{x=x_0} && \text{(chain rule again)}
\end{aligned}$$

Thus, the partial space derivative, and the current, are zero at x_0.

(e) Insufficient.
Again assume the plane wave $\Psi(x,t) = Ae^{ikx - i\omega t}$ with real A and the relation between k and ω, $kx_0 = \omega t_0 + 2n\pi$ for any integer n. This satifies the condition $\Psi(x_0, t_0) = Ae^{ikx_0 - i\omega t_0} = A$.
Followed the same argument in (b), the probability current is then

$$J(x,t) = \frac{\hbar k}{m} A^2$$

for $k = (\omega t_0 + 2n\pi)/x_0$. It is non-zero in general.

Problem 9.14 *Burrowing through walls.*

The details for solving for the tunneling probability through a square barrier are worked out in section 9-5. Using Eq. 9-17 with

$$E = 2 \text{ eV} \qquad V = 4 \text{ eV}$$
$$mc^2 = 0.511 \text{ MeV} \qquad L = 10\mathring{A}$$

we have

$$\alpha = \sqrt{\frac{2mc^2(V-E)}{\hbar^2 c^2}} = \sqrt{\frac{2(0.511 \times 10^6)(2)(1.602 \times 10^{-19})^2}{(1.054 \times 10^{-34})^2 (2.998 \times 10^8)^2}} \text{m}^{-1} \approx 0.7\mathring{A}^{-1}$$

and

$$T \approx 16 \left(\frac{E}{V}\right)\left(1 - \frac{E}{V}\right) e^{-2\alpha L} \approx 4e^{-2(0.7)(10)} \approx 2 \times 10^{-6}.$$

Problem 9.18 *Probability current within a barrier.*

(a) The steady-state wavefunction inside barrier is given by

$$\psi(x) = Be^{-\alpha x} + Ce^{\alpha x} \quad .$$

It's first spatial derivative is then

$$\partial_x \psi(x) = -\alpha Be^{-\alpha x} + \alpha Ce^{\alpha x} \quad .$$

The probability current is then

$$\begin{aligned} J(x,t) &= \frac{-i\hbar}{2m}[(B^* e^{-\alpha x} + C^* e^{\alpha x})(-\alpha Be^{-\alpha x} + \alpha Ce^{\alpha x}) - \\ &\quad (Be^{-\alpha x} + Ce^{\alpha x})(-\alpha B^* e^{-\alpha x} + \alpha C^* e^{\alpha x})] \\ &= \frac{-i\hbar\alpha}{m}(B^* C - BC^*) \quad . \end{aligned}$$

(b) The wavefunction outside the barrier is

$$\psi(x > L) = De^{ikx} \quad .$$

From Eqs.(9-14), the contiunty conditions on ψ and $d\psi/dx$ at $x = L$ are

$$\begin{aligned} De^{ikL} &= Be^{-\alpha L} + Ce^{\alpha L} \\ ikDe^{ikL} &= -\alpha Be^{-\alpha L} + \alpha Ce^{\alpha L} \end{aligned}$$

and the values of coefficients B and C in terms of D are

$$\begin{aligned} B &= \frac{D}{2}\left(1 - \frac{ik}{\alpha}\right) e^{ikL+\alpha L} \\ C &= \frac{D}{2}\left(1 + \frac{ik}{\alpha}\right) e^{ikL-\alpha L} \quad . \end{aligned}$$

Thus, the probability current is

$$\begin{aligned} J(x = L) &= \frac{-i\hbar\alpha}{m}\frac{|D|^2}{4}\left[\left(1 + \frac{ik}{\alpha}\right)^2 - \left(1 - \frac{ik}{\alpha}\right)^2\right] \\ &\quad \frac{\hbar k}{m}|D|^2 \quad . \end{aligned}$$

Problem 10.1 *Slow music.*

A phonograph record weights 50 g and has a radius of 15 cm. It's moment of inertia is

$$I = \frac{1}{2}(0.05)(0.15)^2 \text{ kg m}^2 = 5.625 \times 10^{-4} \text{ kg m}^2.$$

Given that it rotates with $10^9\hbar$ of angular momentum, it's angular velocity is

$$\omega = \frac{L}{I} = \frac{10^9(1.054 \times 10^{-34})}{5.625 \times 10^{-4}} \text{ sec}^{-1} \approx 2 \times 10^{-21} \text{ sec}^{-1}.$$

One revolution time corresponds to

$$T = \frac{2\pi}{\omega} \approx 10^{15} \text{ years}.$$

Angular momentum is quantized in units of $\hbar$. This problem shows that even a billion $\hbar$'s is minuscule with respect to everyday objects.

Problem * *Molecules (two dimensional version).*

Consider a diatomic molecule made up of a heavy atom and a not-so-heavy atom (eg: H_2O, NaI). Let us consider this molecule to be confined to move in two dimensions and allowed only to rotate. For NaI, assume the Iodine atom to be infinitely heavy ($m_I \approx 126 m_H$ where the hydrogen atom mass is $m_H \approx 2 \times 10^{-27}kg$). The sodium atom is roughly $m_{Na} \approx 23 m_H$. For water, let us model it as made of a light H_2 molecule (considered as a single 'atom' with mass $2m_H$) linked to an infinitely heavy O-atom.

(i) *Let us say that the measured spectrum of NaI shows that the longest wavelength in its spectrum has a wavelength of 4cm. Estimate the bond length of the NaI molecule.*

The Schrodinger equation in 2D (r, ϕ) is given by

$$\frac{-\hbar^2}{2\mu}\left(\partial_r^2 + \frac{1}{r^2}\partial_\phi^2\right)\psi + V(\vec{r})\psi = E\psi$$

where the reduced mass μ is essentially the mass of m_{Na} in NaI system or m_{H_2} in simplified H_2O system (as we consider the others as infinite mass). Since both are restricted to be rotating only, we set the reference of the potential at a bond length $r = r_0$ and thus

$$\frac{-\hbar^2}{2\mu r_0^2}\partial_\phi^2\psi = E\psi$$

Thus, the eigenenergy of the rotating system and the corresponding eigenfunction read

$$\begin{aligned} E_m &= \frac{\hbar^2 m^2}{2\mu r_0^2} \\ \psi &= \sqrt{\frac{1}{2\pi}}e^{im\phi} \end{aligned}$$

for any integal quantum number m. The energy spacing between adjacent level is then

$$E_m - E_{m-1} = \frac{\hbar^2(2m-1)}{2\mu r_0^2}$$

The longest-wavelength photon corresponds to the photon with the lowest energy in the spectrum. Its energy matches the smallest difference between the energy spacings, i.e., a $m = 1$ to $m = 0$ transition. Thus,

$$\begin{aligned} \frac{hc}{\lambda} &= \frac{\hbar^2}{2\mu r_0^2} \\ r_0 &= \frac{1}{2\pi}\sqrt{\frac{h\lambda}{2\mu c}} \quad . \end{aligned}$$

Putting $\lambda = 4 \times 10^{-2}m, \mu = m_{Na} \approx 23m_H = 4.6 \times 10^{-26}kg$, $r_0 \approx 1.6 \times 10^{-10}m$.

(ii) *If the H_2-O bond length is 1 Angstrom, estimate the rotational level spacing between the ground and the first excited state (still assuming a two dimensional world). If you want to heat stuff in a microwave, let us assume this is the energy range we want to target. What hole size should the microwave oven shielding have in order to protect you from being cooked when you stand next to the oven waiting for your morning coffee in your two dimensional world?*

Given $r_0 = 10^{-10}m$ and $\mu = m_{H_2} \approx 2m_H = 4 \times 10^{-27}kg$, the spacing betweeen ground state and 1st excited state is

$$E_1 - E_0 = \frac{\hbar^2}{2\mu r_0^2} = 1.4 \times 10^{-22}J = 0.85meV \quad .$$

which corresponds to the photon with wavelength $1.4mm$. Thus, the hole size should not be larger than this wavelength in order to prevent from being cooked!

(iii) *Assume that the NaI molecule is emitted from an oven heated to* $300C$ *for some experiment. Estimate the angular momentum quantum number of NaI in this case.*

The rotational energy E_{rot} is given by

$$E_{rot} = \frac{m^2\hbar^2}{2\mu r_0^2} = 4.8 \times 10^{-24} m^2 \quad ,$$

with the value of r_0 in (i). In temperature $T = 300C$, the thermal energy is $kT \approx 8 \times 10^{-21} J$. The quantum number is therefore

$$m = \sqrt{\frac{8 \times 10^{-21}}{4.8 \times 10^{-24}}} \approx 40.$$