Problem Set 5

PHY 291 H1S

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Problem 5-4: *Energy degeneracy in a cubical box.*

(a) The quantized values of energy for a particle in a cubical box are

$$E = \frac{h^2}{8ma^2}\left(n_x^2 + n_y^2 + n_z^2\right),$$

where n_x, n_y and n_z are positive integers. The lowest energy is

$$E_0 = \frac{3h^2}{8ma^2}$$

with the choice $(n_x, n_y, n_z) = (1, 1, 1)$.

(b) The second-lowest energy level is

$$E_1 = \frac{3h^2}{4ma^2}$$

when the integers (n_x, n_y, n_z) are any of the sets $(2, 1, 1)$, $(1, 2, 1)$ or $(1, 1, 2)$. This energy level is threefold degenerate. Although they have the same energy eigenvalue, the three states are distinguishable: depending on which quantum number takes on the value of 2, there will be an additional node in the wavefunction for this cartesian direction.

(c) The table below gives the number of degenerate states for each value of n^2. When n^2 takes the value 14 we have a sixfold degenerate energy level, i.e. the six possible permutations of the set (1,2,3). (or (1,2,5) for n^2=30, or (3,4,5) for n^2=50, etc.)

n^2	(n_x, n_y, n_z)	degeneracy
3	(1,1,1)	1
6	(1,1,2) (1,2,1) (2,1,1)	3
9	(1,2,2) (2,1,2) (2,2,1)	3
11	(1,1,3) (1,3,1) (3,1,1)	3
12	(2,2,2)	1

Problem 5-6. *Spherically symmetric states in the hydrogen atom.*

(a) We assume a form for our solution,

$$u(r) = (Ar + Br^2)e^{-br},$$

which upon differentiation twice gives

$$\begin{aligned}\frac{d^2}{dr^2}u(r) &= \left[b^2(Ar+Br^2) - 2b(A+2Br) + 2B\right]e^{-br} \\ &= A\left[2\left(\frac{B}{A}-b\right) + \left(b - 4\frac{B}{A}\right)br + \frac{B}{A}(br)^2\right]e^{-br}.\end{aligned}$$

Plugging this into Eq. 5-19 (Schrödinger equation with hydrogen-like potential):

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}u(r) - \frac{Ze^2}{r}u(r) = Eu(r)$$

$$-\frac{\hbar^2}{2m}A\left[2\left(\frac{B}{A}-b\right) + \left(b-4\frac{B}{A}\right)br + \frac{B}{A}(br)^2\right] - \left(\frac{Ze^2}{r}\right)(Ar+Br^2) = E(Ar+Br^2)$$

Equating terms of the same order in r, we have:

$$\begin{aligned}-\frac{\hbar^2}{m}\left(\frac{B}{A}-b\right) - Ze^2 &= 0 && \text{(0th order)} \\ -\frac{\hbar^2}{2m}\left(b-4\frac{B}{A}\right)b - Ze^2\frac{B}{A} &= E && \text{(1st order)} \\ -\frac{\hbar^2 b^2}{2m} &= E && \text{(2nd order)}\end{aligned}$$

From the zeroth-order equation above, we find

$$\frac{B}{A} = b - \frac{Z}{a_0},$$

where

$$a_0 = \frac{\hbar^2}{me^2}.$$

Substituting into the first-order equation:

$$\begin{aligned}-\frac{\hbar^2}{2m}\left(-3b+4\frac{Z}{a_0}\right)b - Ze^2\left(b-\frac{Z}{a_0}\right) &= E \\ \frac{3\hbar^2 b^2}{2m} - 3bZe^2 + \frac{Z^2e^2}{a_0} &= E,\end{aligned}$$

then substracting the second-order equation, we have

$$b^2 - \frac{3}{2}\left(\frac{Z}{a_0}\right)b + \frac{1}{2}\left(\frac{Z}{a_0}\right)^2 = 0.$$

Solving this quadratic polynomial for b gives us two possible solutions, which actually correspond to the two lowest energy states for this potential.

$$\begin{aligned} b &= \frac{Z}{a_0} & \qquad b &= \frac{Z}{2a_0} \\ \frac{B}{A} &= 0 & \qquad \frac{B}{A} &= -\frac{Z}{2a_0} \\ E_1 &= -\frac{\hbar^2 Z^2}{2ma_0^2} & \qquad E_2 &= -\frac{\hbar^2 Z^2}{8ma_0^2} \end{aligned}$$

Problem 5-10. *Normalizing a hydrogenic wavefunction.*

We are seeking the normalized wavefunction for the first excited state of the hydrogen-like atom, with

$$\psi_2(r) = A\left(1 - \frac{Z}{2a_0}r\right)e^{-Zr/2a_0}.$$

Let us begin:

$$\begin{aligned} \int |\psi_2(r)|^2 d\mathbf{r} &= 4\pi|A|^2 \int_0^\infty \left(1 - \frac{Z}{2a_0}r\right)^2 e^{-Zr/a_0} r^2 dr \\ &= 4\pi|A|^2 \int_0^\infty \left(1 - \frac{Z}{a_0}r + \frac{Z^2}{4a_0^2}r^2\right) e^{-Zr/a_0} r^2 dr \\ &= 4\pi|A|^2 \left(I_2 - \frac{Z}{a_0}I_3 + \frac{Z^2}{4a_0^2}I_4\right) \end{aligned}$$

where

$$I_n \equiv \int_0^\infty r^n e^{-Zr/a_0} dr.$$

A quick integration by parts will convince you that the above integral can be calculated recursively:

$$\begin{array}{l} I_n = \frac{a_0}{Z} n I_{n-1} \quad (n \neq 0) \\ I_0 = \frac{a_0}{Z} \end{array} \quad \Rightarrow \quad I_n = n!\left(\tfrac{a_0}{Z}\right)^{n+1}$$

Thus, using the above together with the normalization condition, we have

$$\int |\psi_2(r)|^2 d\mathbf{r} = 8\pi|A|^2\left(\frac{a_0}{Z}\right)^3 = 1$$

and we find $A = \dfrac{1}{2\sqrt{2\pi}}\left(\dfrac{Z}{a_0}\right)^{3/2}$ up to a complex phase.

Problem 5-13. *Expectation values of various quantities.*

(a) To find the expectation value of position of the harmonic oscillator, we need to evaluate

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\psi_n(x)|^2 dx.$$

Since the potential we consider here is symmetric under the mirror transformation $x \to -x$, the wavefunctions $\psi_n(x)$ are either symmetric or antisymmetric: $|\psi_n(x)|^2$ is always even. Thus, our integrand is odd and integrating over $-\infty < x < \infty$ yields an expectation value of zero for any value of the quantum number n.

(d) From table 4-1, the ground state wavefunction is

$$\psi_0(x) = \left(\frac{1}{a\sqrt{\pi}}\right)^{1/2} e^{-x^2/2a^2}.$$

The expectation value is

$$\begin{aligned}\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 |\psi_0(x)|^2 dx \\ &= \frac{1}{a\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/a^2} dx.\end{aligned}$$

Integrating by parts, $\quad u = x \qquad dv = xe^{-x^2/a^2} dx$

$$du = dx \qquad v = -\frac{1}{2}a^2 e^{-x^2/a^2}$$

we get $\langle x^2 \rangle = \dfrac{a}{2\sqrt{\pi}} I(a)$, where $I(a) \equiv \displaystyle\int_{-\infty}^{\infty} e^{-x^2/a^2} dx.$

This is an important integral which we now evaluate. The trick is to take its square, evaluate it with different dummy integration variable, then switch from cartesian to polar coordinates:

$$\begin{aligned}(I(a))^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2/a^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/a^2} dy\right) \\ &= \iint_{-\infty}^{\infty} e^{-(x+y)^2/a^2} dxdy \\ &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2/a^2} r dr d\theta \\ &= 2\pi \left[-\frac{1}{2}a^2 e^{-r^2/a^2}\right]_0^{\infty} \\ &= a^2\pi\end{aligned}$$

Thus, $I(a) = a\sqrt{\pi}$ and $\langle x^2 \rangle = \frac{1}{2}a^2$.

Problem 6-2. *Diagnosis using an xy analyzer.*

(a) Since there is an orientation (call it $\theta = 0$) of the analyzer for which the output of channel x is zero, we conclude that beam A is linearly polarized along y for this orientation. As we rotate the analyzer, intensities from both channels will vary as

$$\begin{aligned} I_x(\theta) &= I_0 \sin^2(\theta), \\ I_y(\theta) &= I_0 \cos^2(\theta). \end{aligned}$$

(b) Since intensities from both channel are independent of the orientation of the analyzer, we conclude that beam B is either circularly polarized or unpolarized.

(c) Beam C is either elliptically polarized or a partial linear polarization.

Problem 7-2. *Combination of quantum amplitudes.*

When the (R) L channel is blocked, the RL analyzer has the effect of projecting onto the (left-) right-circular state. An incident linear state passing through a circular polarizer will see its intensity reduced by half. Transmitted through the RL analyzer is a circularly-polarized beam incident on a y projector, which reduces its intensity by half again. In the end, a quarter of the intensity at A has been transmitted at B.

With both channels open the RL analyzer loop transmits the full intensity of the incident beam and has no effect on its polarization. The y projector completely blocks the transmission of x-linear light; the beam has zero intensity at B.

Using projection amplitude formalism:

(a)
$$\begin{aligned} \text{quantum amplitude } &= \langle y|L\rangle\langle L|x\rangle = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{-i}{\sqrt{2}}\right) = \frac{-i}{2} \\ \text{intensity at B } &= |\langle y|L\rangle\langle L|x\rangle|^2 = \frac{1}{4} \end{aligned}$$

(b)
$$\begin{aligned} \text{quantum amplitude } &= \langle y|R\rangle\langle R|x\rangle = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{i}{\sqrt{2}}\right) = \frac{i}{2} \\ \text{intensity at B } &= |\langle y|R\rangle\langle R|x\rangle|^2 = \frac{1}{4} \end{aligned}$$

(c)
$$\begin{aligned} \text{quantum amplitude } &= \langle y|L\rangle\langle L|x\rangle + \langle y|R\rangle\langle R|x\rangle = \frac{-i}{2} + \frac{i}{2} = 0 \\ \text{intensity at B } &= |\langle y|L\rangle\langle L|x\rangle + \langle y|R\rangle\langle R|x\rangle|^2 = 0 \end{aligned}$$

Problem 7-9. *Properties of a given photon polarization state.*

First, we note that the state $|\psi\rangle$ is correctly normalized, since the projection onto itself gives unity:

$$\begin{aligned}\langle\psi|\psi\rangle &= \left(\frac{3}{5}\langle x| - \frac{4i}{5}\langle y|\right)\left(|x\rangle\frac{3}{5} + |y\rangle\frac{4i}{5}\right)\\ &= \frac{1}{25}\left(9\langle x|x\rangle + 16\langle y|y\rangle + 12i\langle x|y\rangle - 12i\langle y|x\rangle\right)\\ &= 1\end{aligned}$$

If this wasn't the case, we would have to renormalize our results by dividing by $|\langle\psi|\psi\rangle|^2$ everywhere.

(a) The quantum amplitude for projecting along y is $\langle y|\psi\rangle = \frac{4i}{5}$. The fractions of photon passing a y projector is

$$|\langle y|\psi\rangle|^2 = \frac{16}{25} = 64\%.$$

(b) Projecting along x' corresponds to the amplitude $\langle x'|\psi\rangle = \frac{3}{5}\cos\theta + \frac{4i}{5}\sin\theta$. The fraction of transmitted photon is

$$|\langle x'|\psi\rangle|^2 = \frac{9}{25}\cos^2\theta + \frac{16}{25}\sin^2\theta = \frac{1}{25}\left(9 + 7\sin^2\theta\right).$$

(c) We need to identify how much of the beam is circularly polarized. The quantum amplitude and probability for projecting into the left- and right-circular states are

$$\begin{aligned}\langle L|\psi\rangle &= \frac{3}{5}\frac{-i}{\sqrt{2}} + \frac{4i}{5}\frac{1}{\sqrt{2}} = \frac{i}{5\sqrt{2}} &&\Rightarrow |\langle L|\psi\rangle|^2 = \frac{1}{50},\\ \langle R|\psi\rangle &= \frac{3}{5}\frac{i}{\sqrt{2}} + \frac{4i}{5}\frac{1}{\sqrt{2}} = \frac{7i}{5\sqrt{2}} &&\Rightarrow |\langle R|\psi\rangle|^2 = \frac{49}{50}.\end{aligned}$$

This indicates that a photon in the state described by $|\psi\rangle$ is found to be left- (right-) circularly polarized 2% (98%) of the time. A left- (right-) polarized photon carries $\hbar$ ($-\hbar$) of angular momentum. On average, the angular momentum added to the absorbing surface is

$$\frac{1}{50}(\hbar) + \frac{49}{50}(-\hbar) = -\frac{24}{25}\hbar$$

per photon.