Problem Set 4

PHY 291 H1S

February 13, 2008

Problem 4-5: *The consequences of symmetry.*

(a) We know that $\psi_n(x)$ is a solution to Schrödinger's Equation, such that

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_n(x)}{\partial x^2}+V(x)\psi_n(x)=E_n\psi_n(x).$$

If we reflect everything about the $x=0$ axis (as though we reflect the system in a mirror), it will obey exactly the same physics with $x\to -x$, that is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_n(-x)}{\partial(-x)^2}+V(-x)\psi_n(-x)=E_n\psi_n(-x).$$

We are given that $V(-x)=V(x)$. Since this is the case, we may say that there is another solution to the problem, $\phi(x)=\psi(-x)$:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi_n(x)}{\partial x^2}+V(x)\phi_n(x)=E_n\phi_n(x).$$

As this matches the first Schrödinger equation with $\phi(x)\to\psi(x)$, $\phi(x)$ is also a solution.

(b) Assume there are two distinct eigenfunctions, i.e., ϕ_1 and ϕ_2 which are corresponding to same eigenvalue E_n. The SE of each eigenfunction are given by

$$\begin{aligned}-\frac{\hbar^2}{2m}d_x^2\phi_1(x)+V(x)\phi_1(x)&=E_n\phi_1(x),\\ -\frac{\hbar^2}{2m}d_x^2\phi_2(x)+V(x)\phi_2(x)&=E_n\phi_2(x).\end{aligned}$$

Mutliplying each expression by another eigenfunctions, and substract them together, thus gives

$$\phi_2 d_x^2\phi_1-\phi_1 d_x^2\phi_2=0.$$

Integrate the above in all x-space, integrating by parts, we then have

$$\phi_2 d_x\phi_1-\phi_1 d_x\phi_2=0.$$

Shuffle the expression and we get

$$\frac{1}{\phi_1}\frac{d\phi_1}{dx}=\frac{1}{\phi_2}\frac{d\phi_2}{dx}.$$

Integrate it once more over x and we have

$$\ln \phi_1 = \ln \phi_2 + C,$$

i.e., $\phi_1 = C'\phi_2$ where C and C' are constants. Thus, there is only one eigenfunction corresponding to each eigen-energy in 1-d case.

(c) Following the previous observations, since $\psi_n(-x)$ and $\psi_n(x)$ share the same eigenvalue, it should be that

$$\psi_n(-x) = \alpha\psi_n(x)$$

for some constant α. Also, $|\alpha|^2$ must be unity in order to retain normalization. Thus, α lies on the unit circle in the complex plane ($\alpha = e^{i\theta}$ with $0 \le \theta < 2\pi$). For 1-d potentials, we can always choose the wavefunction to be real. There are two values of θ which guarantee the reality of α: $\theta = \{0, \pi\}$. So $\alpha = \pm 1$, and $\psi_n(-x) = \pm\psi_n(x)$.

Problem 4-6. *The simple harmonic oscillator.*

(a) The width parameter is $a = (\hbar/m\omega_0)^{1/2}$. The maximum displacement of the classical oscillator is given by the condition that the total energy is equal to the potential energy $V(x) = \frac{1}{2}Cx^2 = \frac{1}{2}m\omega_0^2x^2$. If the total energy is $\frac{1}{2}\hbar\omega_0$, then

$$\frac{1}{2}\hbar\omega_0 = \frac{1}{2}m\omega_0^2x_{max}^2$$

$$x_{\max} = \sqrt{\frac{\hbar}{m\omega_0}}$$

(b) We take $\psi(x) = (1 + bx^2)e^{-x^2/2a^2}$ and plug it into Schrödinger's equation with $V(x) = \frac{1}{2}m\omega_0^2x^2$:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + \frac{1}{2}m\omega_0^2x^2\psi(x) = E_n\psi(x).$$

The first derivative of $\psi(x)$:

$$\frac{\partial\psi(x)}{\partial x} = 2bxe^{-x^2/2a^2} + (1 + bx^2)(\frac{-x}{a^2})e^{-x^2/2a^2}.$$

The second derivative of $\psi(x)$:

$$\begin{aligned}
\frac{\partial^2\psi(x)}{\partial x^2} &= 2be^{-x^2/2a^2} + 2bx(\frac{-x}{a^2})e^{-x^2/2a^2} + 2bx(\frac{-x}{a^2})e^{-x^2/2a^2} \\
&\quad +(1+bx^2)(\frac{-1}{a^2})(\frac{-x}{a^2})e^{-x^2/2a^2} + (1+bx^2)(\frac{-x}{a^2})^2e^{-x^2/2a^2} \\
&= e^{-x^2/2a^2}\left[2b + 2bx(\frac{-x}{a^2}) + 2bx(\frac{-x}{a^2}) + (1+bx^2)(\frac{-1}{a^2})(\frac{-x}{a^2}) + (1+bx^2)(\frac{-x}{a^2})^2\right] \\
&= e^{-x^2/2a^2}\left[bx^2\left(-\frac{4}{a^2} + \frac{x^2}{a^4} - \frac{1}{a^2}\right) + 1\left(2b + \frac{x^2}{a^4} - \frac{1}{a^2}\right)\right]
\end{aligned}$$

We notice that the quantities in round brackets are very similar, and are, indeed, the same if we assign $2b = -4/a^2$ or $b = -2/a^2$. In this case,

$$\begin{aligned}
\frac{\partial^2\psi(x)}{\partial x^2} &= e^{-x^2/2a^2}\left[(1+bx^2)\left(\frac{x^2}{a^4} - \frac{5}{a^2}\right)\right] \\
&= \psi(x)\left(\frac{x^2}{a^4} - \frac{5}{a^2}\right)
\end{aligned}$$

Putting this into the Schrödinger equation, then substituting for $a^2 = \hbar/m\omega_0$

$$\begin{aligned}
\frac{-\hbar^2}{2m}\left(\frac{x^2}{a^4} - \frac{5}{a^2}\right)\psi(x) + \frac{1}{2}m\omega_0^2x^2\psi(x) &= E_n\psi(x) \\
\frac{-\hbar^2}{2m}\left(\frac{x^2}{a^4} - \frac{5}{a^2}\right) + \frac{1}{2}m\omega_0^2x^2 &= E_n \\
\frac{-\hbar^2}{2m}\left(\frac{x^2m^2\omega_0^2}{\hbar^2} - \frac{5m\omega_0}{\hbar}\right) + \frac{1}{2}m\omega_0^2x^2 &= E_n \\
-\frac{1}{2}m\omega_0^2x^2 + \frac{5}{2}\hbar\omega_0 + \frac{1}{2}m\omega_0^2x^2 &= E_n \\
E_n &= \frac{5}{2}\hbar\omega_0
\end{aligned}$$

Thus, this expression satisfies Schrödinger's equation for $E = \frac{5}{2}\hbar\omega_0$. We require that $b = -2/a^2$.

Problem 4-9. *Classically forbidden regions for the simple harmonic oscillator.*

The normalized wavefunction for the ground state of the one-dimensional simple harmonic oscillator is given in Table 4-1:

$$\psi_0(x) = \left(\frac{1}{a\sqrt{\pi}}\right)^{1/2} e^{-x^2/2a^2}$$

As we saw in Problem 4-6, the classically allowed region falls between a and $-a$. We integrate the probability of finding the particle outside this region:

$$\begin{aligned} P(|x| > a) &= 2\int_a^\infty |\psi(x)|^2 dx \\ &= \frac{2}{a\sqrt{\pi}} \int_a^\infty e^{-x^2/a^2} dx \end{aligned}$$

We make a substitution $t = x/a$, with $dt = dx/a$

$$P(|x| > a) = \frac{2}{\sqrt{\pi}} \int_1^\infty e^{-t^2} dt$$

This is a standard (non-analytic) integral called the complimentary error function, often denoted by "erfc" in mathematics programs. For example, in MATLAB, erfc is defined as "erfc(x) = 2/sqrt(pi) * integral from x to inf of exp(-t∧2) dt = 1 - erf(x)" by typing "help erfc" at the command line. Thus, if we ask MATLAB for the value of our complimentary error function, we have

$$P(|x| > a) = \text{erfc}(1) \approx 0.157$$

So, we find the particle outside the classically allowed region about 16% of the time.

Problem 4-10. *Visual observation of a quantum oscillator.*

(a) To find the approximate quantum number of the system, we determine its energy and divide by the energy per quantum. Since we have been given that the amplitude of oscillation is 10^{-5} m, we can determine energy from x_{max}, as in Problem 4-6. The total energy is:

$$\begin{aligned} E_{\text{total}} &= \frac{1}{2} m\omega_0^2 x_{max}^2 \\ &= \frac{1}{2}(10^{-9}\text{kg})(2\pi \times 1000\text{Hz})^2(10^{-5}\text{m})^2 \\ &= 1.97 \times 10^{-12}\text{J} \end{aligned}$$

while a single quantum of energy is

$$\begin{aligned} E_{\text{quantum}} &= h\nu_0 \\ &= (6.63 \times 10^{-34}\text{Js})(1000\text{Hz}) \\ &= 6.63 \times 10^{-31}\text{J} \end{aligned}$$

Thus, we can calculate the approximate quantum number:

$$N_{quanta} = \frac{E_{\text{total}}}{E_{\text{quantum}}} \quad = \quad 3 \times 10^{18}$$

(b) In the lowest energy state, $E = \frac{1}{2}h\nu_0$,

$$\begin{aligned} E_0 &= \frac{1}{2}(6.63 \times 10^{-34}\text{Js})(1000\text{Hz})/(1.6 \times 10^{-19}\text{eV/J}) \\ &= 2.1 \times 10^{-12}\text{eV} \end{aligned}$$

which is about 10 orders of magnitude smaller than average thermal energy of air molecules at room temperature (1/40eV = 0.025eV).

(c) The classical amplitude of vibration in the lowest energy state is given by $a = \sqrt{(\hbar/m\omega_0)}$.

$$\begin{aligned} a &= \left(\frac{(6.63 \times 10^{-34}/2\pi\text{Js})}{(10^{-9}\text{kg})(2\pi \times 1000\text{Hz})}\right)^{1/2} \\ &= 4.1 \times 10^{-15}\text{m} \end{aligned}$$

which is about 8 orders of magnitude smaller than the wavelength of visible light ($\lambda \approx 500$nm), which means that it is impossible to optically observe a vibration of this amplitude.

(d) As a referree, I would not award a grant to carry out this proposal, as the scales on which any visual observation could be made is impossible due to the smallness of the amplitude of single quantum vibrations and the thermal noise that would overwhelm the system.