UNIVERSITY OF TORONTO

Test, Mar 31, 2008

PHY291S: ELEMENTS OF PHYSICS III (Quantum Physics)

Aid: Open Book, Open Notes, Any Calculator Allowed

Duration of exam: 60 mins (10am-11am)

Instructions:

1. There are five questions in this test, five pages including this cover page and rough work page at the end if needed.
2. Answer all questions.
3. You do not need to show all steps, you will be marked on the final answer only.
4. PLEASE WRITE YOUR NAME AND STUDENT NUMBER ON EACH PAGE OF THE EXAM BOOK BEFORE YOU BEGIN.
5. If needed assume: Electron mass $m_e = 10^{-30} kg$, $\hbar = 10^{-34} Jsec$.

NAME:

STUDENT NUMBER:

1. You are given that the ground state wavefunction of a Hamiltonian $H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_0(x)$ is $\Psi_0(x)$, and the ground state energy is E_0. Which of the following statements is true about the ground state wavefunction $\Psi_1(x)$ of a new Hamiltonian $H_1 = 2H_0 + V_1$, where V_1 is independent of x? (1 point)

(a) $\Psi_1(x) = \Psi_0^2(x)$.

→ (b) $\Psi_1(x) = \Psi_0(x)$.

(c) $\Psi_1(x) = 2\Psi_0(x) + (V_0(x) - V_1)/E_0$.

(d) Cannot be determined from the available information.

If $H_1 = 2H_0 + V$

then Schrödinger's Eqn reads $H_1 \Psi_1(x) = E_1 \Psi_1(x)$

$\rightarrow 2H_0 \Psi_1(x) + V_1 \Psi_1(x) = E_1 \Psi_1(x)$

$2H_0 \Psi_1(x) = (E_1 - V_1)\Psi_1(x)$

(V_1 creates an offset to the eigenvalue).

$H_0 \Psi_1(x) = \frac{(E_1 - V_1)}{2} \Psi_1(x)$

But $H_0 \Psi_0(x) = E_0 \Psi_0(x) \Rightarrow \Psi_1(x) = \Psi_0(x)$

They solve 2 Schrödinger Eqns with the same x-dependence, but different eigenvalues.

2. At time $t = 0$, you are given a superposition state $\exp(ikx) + \exp(-ikx)$ describing free particles. Is this a stationary state? Why or why not? (1 point)

$$\Psi(x,t) = A\left(e^{ikx - i\omega t} + e^{-ikx - i\omega t}\right)$$

$$= A\left(e^{ikx} + e^{-ikx}\right) e^{-i\omega t}$$

↳ ω is the same for both the right- and left-travelling components. $\left(\omega = \frac{E}{\hbar} = \frac{\hbar k^2}{2m}\right)$

Probability $= |\Psi(x,t)|^2$ has no time dependence

$\Rightarrow \boxed{\text{stationary state}}$

3. At time $t = 0$, you have a superposition state

$$\psi(x) = A[\sin(\pi x/L) + 3\sin(2\pi x/L)] \tag{1}$$

for a particle in a box with walls at $x = 0, L$.

a) Is this a stationary state? (1 point)

b) What is the normalization constant A? (1 point)

c) If the right wall is suddenly moved from $x = L$ to $x = 2L$ at time $t = 0$, what is the probability at that instant that it is in the first excited state (n=2 state) of the new box? (2 points)

a) This is <u>not</u> a stationary state.

$$\Psi(x,t) = A\left[\sin\left(\frac{\pi x}{L}\right)e^{-iE_1t/\hbar} + 3\sin\left(\frac{2\pi x}{L}\right)e^{-iE_2t/\hbar}\right]$$

since $E_1 \neq E_2$, this state evolves in time.

b) $1 = \int_{-\infty}^{\infty} \psi^*(x)\psi(x)\,dx$

rewrite $\psi(x) = A\left[\sqrt{\frac{L}{2}}\psi_1(x) + 3\sqrt{\frac{L}{2}}\psi_2(x)\right]$

where $\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$

and $\int_{-\infty}^{\infty}\psi_n^*(x)\psi_m(x)\,dx = \begin{cases} 1 & m=n \\ 0 & m\neq n\end{cases}$

(orthonormality of eigenstates).

$$1 = |A|^2\left(\frac{L}{2}\right)\int_{-\infty}^{\infty}\left[\psi_1^*(x)\psi_1(x) + 3\psi_1^*(x)\psi_2(x) + 3\psi_2^*(x)\psi_1(x) + 9|\psi_2(x)|^2\right]dx$$

$$= |A|^2\left(\frac{L}{2}\right)\left[1 + 0 + 0 + 9\right]$$

$$= |A|^2 5L$$

$$\boxed{A = \pm\sqrt{\frac{1}{5L}}}$$

c) (following page)

c) At the first instant, the wavefunction in the new box is: $\psi(x) = \sqrt{\frac{1}{5L}}\left[\sin\left(\frac{\pi x}{L}\right) + 3\sin\left(\frac{2\pi x}{L}\right)\right] \quad x \in [0, L]$.

The eigenstates of the new expanded well are:

$$\psi_n = \sqrt{\frac{2}{(2L)}}\,\sin\left(\frac{n\pi x}{(2L)}\right)$$

ie, the eigenstates of the infinite square well of width '2L'.

$$\psi_n = \sqrt{\frac{1}{L}}\,\sin\left(\frac{n\pi x}{2L}\right). \quad 0 \le x \le 2L$$

The second excited state of the new well is $\psi_2(x) = \sqrt{\frac{1}{L}}\sin\left(\frac{\pi x}{L}\right)$ $0 \le x \le 2L$.

We can find the probability of being in the ~~2nd~~ $n=2$ state by taking the coefficient B_n, redefined for a well of width $2L$ (Eq. 8-9, French + Taylor)

$$B_n = \frac{2}{2L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{2L}\right)dx$$

where $f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2L}\right)$

$$= \sqrt{L}\sum_{n=1}^{\infty} B_n\psi_n(x)$$

such that the probability of finding the state $f(x)$ in n is $|B_n\sqrt{L}|^2 = P_n$

So, $$B_2 = \frac{1}{L}\int_0^{2L}\underbrace{\frac{1}{\sqrt{5L}}\left(\sin\left(\frac{\pi x}{L}\right) + 3\sin\left(\frac{2\pi x}{L}\right)\right)}_{\text{only defined on } 0 \le x \le L}\sin\left(\frac{\pi x}{L}\right)dx$$

$\Rightarrow$ lims of integral $\to (0, L)$

$$B_2 = \frac{1}{L\sqrt{5L}}\int_0^L\left[\sin^2\left(\frac{\pi x}{L}\right) + 3\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right)\right]dx$$

$$= \frac{1}{L\sqrt{5L}}\left[\frac{L}{2} + 3(0)\right]$$ [Eq. 8-8, result on p 325 French + Taylor]

$$= \frac{1}{2\sqrt{5L}}$$

$$P(2) = |B_2\sqrt{L}|^2 = \left|\frac{1}{2\sqrt{5}}\right|^2 = \boxed{\frac{1}{20} = P_2}$$

4. Po^{212}, a radioactive isotope of Polonium, emits alpha particles with an energy of about $10MeV$. The half life of Po^{212} is 5×10^{-7} sec. Find the spread in energy of the emitted alpha particles. (2 points)

$$\Delta E \, \Delta t \sim \hbar$$

$$\Delta E \sim \frac{\hbar}{\Delta t} \sim \frac{10^{-34}\ \mathrm{J}\cdot \cancel{\mathrm{s}}}{5\times 10^{-7}\ \cancel{\mathrm{s}}} = \frac{1}{5}\times 10^{-27}$$

$$\boxed{\Delta E = 2\times 10^{-28}\ \mathrm{J}}$$

5. An electron has kinetic energy of 9.0 eV. It is incident upon a rectangular barrier of height 13.0 eV and thickness 0.3 nm. What is the probability for this electron to tunnel across the barrier? (2 points)

There are several accepted solutions, depending on the approximation made.

For solutions inside barrier, $\psi_{II}(x) = Be^{-\alpha x} + Ce^{\alpha x}$

where $\alpha = \dfrac{\sqrt{2m(V-E)}}{\hbar}$

If $\alpha L \gg 1$, $T \simeq 16\left(\frac{E}{V}\right)\left(1-\frac{E}{V}\right)e^{-2\alpha L}$ (Eq. 9-17, French+Taylor)

$$\alpha L = \frac{\sqrt{2m_e(13-9)\mathrm{eV}}}{\hbar}L = \frac{\sqrt{2(10^{-30}\mathrm{kg})(13-9)(1.6\times10^{-19}\mathrm{J})}}{10^{-34}\mathrm{J\cdot s}}(0.3\times10^{-9})\mathrm{m}$$

$= 3.39$ (we'll assume this is enough bigger than 1).

$$T \simeq 16\left(\frac{9}{13}\right)\left(1-\frac{9}{13}\right)e^{-2(3.39)} = \boxed{3.8\times10^{-3} \approx T}$$

Alternate approx → $T \approx e^{-2\alpha} = e^{-2(3.39)} =$ ~~[illegible]~~

$$\boxed{T \approx 1.1\times10^{-3}}$$