Suggested Solution

Name: Student number:

University of Toronto
PHY291
Winter, 2008 Midterm Exam

Duration – 1.5 hours, No aids allowed

Instructions:

- **DON'T FORGET TO ENTER YOUR NAME AND STUDENT NUMBER ABOVE!!!**
- Answer all five problems in the space provided.
- In all cases, show your work. The answer is worth **0** if you don't.
- Due to your lack of a calculator, assume:
  $c = 3\text{x}10^8$m/s
  $h = 7\text{x}10^{-34}$Js = $4\text{x}10^{-15}$eV s
  $\pi = 3$
  $\hbar = 1\text{x}10^{-34}$Js
  $m_{electron} = 1\text{x}10^{-30}$kg
- Attempt to avoid panicking, except insofar as you feel it will improve your grade.
- $\frac{-\hbar^2}{2\text{m}}\nabla^2\psi + V\psi = E\psi$
- $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$ and $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$

Question 1	/12
Question 2	/12
Question 3	/12
Question 4	/12
Question 5	/12
Total	/60

Problem 1: Photons and bolometric detectors.

A bolometric detector measures the amount of energy striking it in each time period. Imagine that 300nm radiation strikes a cooled bolometric detector.

a) What is the frequency of the radiation?

$$E = h\nu \qquad \nu = \frac{c}{\lambda} = \frac{3\times10^{8}\,m/s}{3\times10^{-7}\,m} = \boxed{1\times10^{15}\,Hz = \nu}$$

b) How much energy is there in one photon?

$$E = h\nu = 7\times10^{-34}\,J\cdot s \times 1\times10^{15}\,Hz = \boxed{7\times10^{-19}\,J = E}$$

c) Imagine that the detector has an *rms* noise (per uncorrelated 1ms sample) of 3.5×10^{-20}J. What is the signal to noise per photon? Can this detector see individual photons?

$$\frac{S}{N} = \frac{7\times10^{-19}}{3.5\times10^{-20}} = 2\times10^{1} = \boxed{20 = S/N}$$

Yes, this detector should detect single photons.

d) Could this detector be used to determine (approximately) the wavelength of individual photons hitting it (assume that they are near, but not necessarily exactly 300 nm)? Would the brightness of the light effect your answer? Explain.

Yes, you could measure the energy of single photons hitting the detector, and calculate $E = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{E}$

The brightness will affect the answer → if there is more than one photon at a time, the procedure above does not work.

Problem 2: Electron Microscope

The target resolution of an electron microscope is 0.8 angstroms.
a) What about an electron limits the resolution of an electron microscope?

The wavelength (de Broglie) of the electron limits the resolution. $\lambda_{dB} = \frac{h}{p}$.

b) What voltage would be required for accelerating the electrons in order to achieve this resolution?

From de Broglie relation, $p = \frac{h}{\lambda}$

Check if the accelerated e^- is in relativistic regime:

$$pc = \frac{hc}{\lambda} = 2.3 \times 10^{-15}\ J$$

$$m_e c^2 = 10^{-30} \times (3 \times 10^8)^2 \approx 9 \times 10^{-14} J \gg pc$$

$\therefore$ The accelerated e^- is in non-relativistic regime.

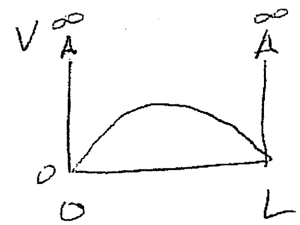
Energy supplied $E = eV$

$$= \frac{p^2}{2m_e} \Rightarrow V = \frac{(h/\lambda)^2}{2m_e e}$$

$\therefore$ Voltage required $V = 235\ V$ (or $175.5V - 262.3\ V$)

Problem 3: Normalization of a wavefunction:

A particle is in the ground state of an infinite square well potential of width L:



$V(x) = 0 \qquad 0 < x < L$
$V(x) = \infty \qquad 0 > x \text{ or } x > L$

so,

$\psi(x) = A\sin(\pi x/L) \qquad 0 < x < L$
$\psi(x) = 0 \qquad 0 > x \text{ or } x > L$

a) What is A so that the wave function is normalized?

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 dx$$
$$= A^2 \int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx$$
$$= A^2 \int_0^L \left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi x}{L}\right)\right] dx$$
$$= A^2 \left[\frac{1}{2}x\Big|_0^L - \frac{L}{\pi}\sin\left(\frac{2\pi x}{L}\right)\Big|_0^L\right]$$

$$1 = A^2\left(\frac{L}{2}\right)$$
$$\boxed{A = \sqrt{\frac{2}{L}}}$$

b) What is the probability that, in an experiment, the particle would be found between $L/4$ and $L/2$?

$$P(L/4 \to L/2) = \int_{L/4}^{L/2} |\psi(x)|^2 dx$$
$$= \frac{2}{L}\int_{L/4}^{L/2} \left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi x}{L}\right)\right] dx$$
$$= \frac{2}{L}\left[\frac{1}{2}x\Big|_{L/4}^{L/2} - \frac{L}{4\pi}\sin\left(\frac{2\pi x}{L}\right)\Big|_{L/4}^{L/2}\right]$$

$$P = \frac{2}{L}\left[\frac{1}{2}\left(\frac{L}{2} - \frac{L}{4}\right) - \frac{L}{4\pi}\sin(\pi) + \frac{L}{4\pi}\sin\left(\frac{\pi}{2}\right)\right]$$
$$= \frac{2}{L}\left(\frac{L}{8} + \frac{L}{4\pi}\right)$$
$$\boxed{P = \frac{1}{4} + \frac{1}{2\pi}}$$
$$\left(\frac{1}{4} + \frac{1}{2\pi}\right)$$

c) What energy does the particle have (in terms of L and fundamental constants)?

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

$$\frac{d}{dx}\psi(x) = \frac{\pi}{L} A\cos\left(\frac{\pi x}{L}\right)$$

$$\frac{d^2}{dx^2} = -\frac{\pi^2}{L^2} A\sin\left(\frac{\pi x}{L}\right) = -\frac{\pi^2}{L^2}\psi(x)$$

$$E\cancel{\psi(x)} = -\frac{\hbar^2}{2m}\left(\frac{-\pi^2}{L^2}\right)\cancel{\psi(x)} + 0\,\psi(x)$$

$$\boxed{E = \frac{\hbar^2\pi^2}{2mL^2} = \frac{h^2}{8mL^2}}$$

Problem 4: Expectation Values

For the particle in the infinite square well potential from problem 3, calculate the following expectation values, in terms of L and fundamental constants.

a) <x>: Position

$$\langle x\rangle = \int_{-\infty}^{\infty} x\,|\psi(x)|^2\,dx = \int_0^L A^2 \sin^2\left(\frac{\pi x}{L}\right) x\,dx$$

$$\boxed{\langle x\rangle = \frac{L}{2}}$$

$$= \frac{2}{L}\int_0^L \frac{1}{2}x - \frac{1}{2}x\cos\left(\frac{2\pi x}{L}\right)dx$$

$$= \frac{2}{L}\left[\frac{L^2}{4} - \frac{1}{2}\frac{Lx}{2\pi}\sin\left(\frac{2\pi x}{L}\right)\Big|_0^L + \frac{1}{2}\int_0^L \frac{L}{2\pi}\underbrace{\cos\left(\frac{2\pi x}{L}\right)dx}_{0}\right]$$

b) $\langle(x-L/2)^2\rangle$: Squared separation from the center.

$$\left\langle\left(x-\frac{L}{2}\right)^2\right\rangle = \int_{-\infty}^{\infty}\psi^*(x)\left(x-\frac{L}{2}\right)^2\psi(x)\,dx$$

$$= A^2\int_0^L\left(x^2 - xL + \frac{L^2}{4}\right)\sin^2\left(\frac{\pi x}{L}\right)dx$$

$$= \frac{2}{L}\left[\underbrace{\int_0^L x^2\sin^2\left(\frac{\pi x}{L}\right)dx}_{I} - \underbrace{\int_0^L xL\sin^2\left(\frac{\pi x}{L}\right)dx}_{L\cdot\frac{L^2}{4}} + \underbrace{\frac{L^2}{4}\int_0^L\sin^2\left(\frac{\pi x}{L}\right)dx}_{=\frac{L}{2}\,(3a)}\right]$$

$$-\frac{L^3}{4} + \frac{L^3}{8} = -\frac{L^3}{8}$$

$$I = \int_0^L x^2\sin^2\left(\frac{\pi x}{L}\right)dx = \int_0^L\left[\frac{1}{2}x^2 - \underbrace{\frac{1}{2}x^2\cos\left(\frac{2\pi x}{L}\right)}_{I_2}\right]dx$$

$$I_2 = \int_0^L \frac{1}{2}x^2\cos\left(\frac{2\pi x}{L}\right)dx \qquad u = x^2 \quad dv = \cos\left(\frac{2\pi x}{L}\right) \qquad du = 2x\,dx \quad v = \frac{L}{2\pi}\sin\left(\frac{2\pi x}{L}\right)$$

$$= \frac{1}{2}\frac{Lx^2}{2\pi}\underbrace{\sin\left(\frac{2\pi x}{L}\right)\Big|_0^L}_{0} - \frac{L}{2\pi}\int_0^L x\sin\left(\frac{2\pi x}{L}\right)dx \qquad u = x \quad dv = \sin\left(\frac{2\pi x}{L}\right) \qquad du = dx \quad v = -\frac{L}{2\pi}\cos\left(\frac{2\pi x}{L}\right)$$

$$= -\left(\frac{L}{2\pi}\right)\left[-\frac{Lx}{2\pi}\cos\left(\frac{2\pi x}{L}\right)\Big|_0^L + \frac{L}{2\pi}\int_0^L \underbrace{\cos\left(\frac{2\pi x}{L}\right)dx}_{0}\right]$$

$$= \frac{L^3}{(2\pi)^2}$$

$$I = \frac{L^3}{6} - \frac{L^3}{(2\pi)^2}$$

$$\left\langle\left(x-\frac{L}{2}\right)^2\right\rangle = \frac{2}{L}\left[\frac{L^3}{6} - \frac{L^3}{(2\pi)^2} - \frac{L^3}{4} + \frac{L^3}{8}\right] = L^2\left(\frac{1}{3} - \frac{1}{2\pi^2} + \frac{1}{4}\right) = \boxed{L^2\left(\frac{1}{12} - \frac{1}{2\pi^2}\right)}$$

$$= L^2\left(\frac{1}{12} - \frac{1}{2\pi^2}\right).$$

Problem 5: Polarization Quantum State:

A beam of photons traveling in the z direction are in a polarization state described by

$$|\psi\rangle = |x\rangle 4/5 + |y\rangle 3i/5$$

What fraction of photons will pass through a polarizer rotated by an angle θ relative to the x axis.

polarizer at θ: project onto a vector $|x'\rangle$

such that $\langle x'|x\rangle = \cos\theta$

$\langle x'|y\rangle = \sin\theta$.

$$\langle x'|\psi\rangle = \langle x'|x\rangle \tfrac{4}{5} + \langle x'|y\rangle \frac{3i}{5}$$

$$= \frac{4}{5}\cos\theta + \frac{3i}{5}\sin\theta.$$

$$|\langle x'|\psi\rangle|^2 = \left(\frac{4}{5}\cos\theta\right)^2 + \left(\frac{3\sin\theta}{5}\right)^2$$

$$= \frac{16}{25}\cos^2\theta + \frac{9}{25}\sin^2\theta$$

$$\boxed{P(\text{passing}) = \frac{9}{25} + \frac{7}{25}\cos^2\theta.}$$