MAT194F Calculus
Midterm Test No. 1
9:00 – 10:45, 16 October 2008
105 minutes
No calculators or aids

1. **[10 marks]**. Express the limit as a definite integral:

   (a) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{(3/n)}{(1+(i/n)^4)}$

   (b) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{(3/n)}{(1+i/n)^4}$

2. **[10 marks]**. Use the method of substitution to evaluate:

   (a) $\int_1^2 x\sqrt{x-1}\,dx$

   (b) $\int_{1/2}^{1}\frac{\cos(x^{-2})}{x^3}dx$

3. **[10 marks]**. Consider the function $y = h(x) = 3x + 1$. Describe and draw a picture of the set $\{x: 1 < |h(x) - 7| < 2\}$.

4. **[10 marks]**. Let $f(x)=\begin{cases}1, & 0\le x\le 1\\ -1, & 1< x\le 2\\ 2, & 2< x\le 3\end{cases}$

   For $0\le x\le 3$ define $F(x)=\int_0^x f(s)ds$. Plot the graphs of $f(x)$ and $F(x)$.

5. **[10 marks]**. (a) What x is specified by $\frac{2x-1}{x+3}>2$?

   (b) Find all x that satisfy $|x - 3| > |-x + 2|$ purely from algebraic arguments.

   Then also show using a sketch that your answer is correct geometrically.

6. [**10 marks**]. (a) If $g(x)=\begin{cases} x^2, \text{if x is rational} \\ 0, \text{ if x is irrational} \end{cases}$

prove that $\lim_{x \to 0} g(x) = 0$.

(b) Prove also that $\lim_{x \to 1} g(x)$ ~~$=0$~~ does not exist.

7. [**20 marks**]. Provide a rigorous proof, such as "$\delta - \varepsilon$ proof", that:

(a) $\lim_{x \to 0} x^2 = 0$

(b) $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

8. [**20 marks**]. Use the basic definition of the derivative to prove that:

(a) $\frac{d}{dx}(x^2) = 2x$

(b) $\frac{d}{dx}(\frac{1}{x}) = -\frac{1}{x^2}$

(c) $\frac{d}{dx}(\frac{1}{\sqrt{x}}) = -\frac{1}{2}x^{-3/2}$

Then confirm your answers by applying the Fundamental Theorem of Calculus.

MAT194F Calculus
Midterm Test No. 2
9:00 – 10:45, 20 November 2008
105 minutes
No calculators or aids

1. **[10 marks]** Suppose f is twice continuous (and differentiable) on $\Re$ (real numbers) and has 3 roots. Show that f'' has at least one root.

2. **[10 marks]** The linear density of a rod 6 m long is $5/\sqrt{2x+1}$ kg/m, where x is measured in meters from one end of the rod. Find the average density of the rod.

3. **[10 marks]** If f is continuous on [0, 1] prove that $\int_0^1 f(x)dx = \int_0^1 f(1-x)dx$.

4. **[10 marks]** Find the volume of the solid obtained by rotating the region bounded by the curves $y = x^2$ and $x = y^2$ about the line $x = -1$.

5. **[10 marks]** Determine if $f'(0)$ exists for:

   (a) $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, x \neq 0 \\ 0, x = 0 \end{cases}$ (b) $f(x) = \begin{cases} x\sin\frac{1}{x^2}, x \neq 0 \\ 0, x = 0 \end{cases}$

6. **[10 marks]** Sketch $f(x) = x^{7/5} - \frac{7}{2}x^{2/5}$ showing all important information.

7. **[10 marks]** A figure is represented by the equation $x^4 - xy + y^4 = 1$. Find the points at which this figure crosses the x-axis and show that the tangent lines at these points are parallel.

8. **[10 marks]** Prove that $|\cos a - \cos b| \leq |a - b|$ for all $a, b \in \Re$.

9. (a) [**4 marks**] Two circles lie in a plane. The circle of radius 1 meter overlaps the circle of smaller radius r in such a way that their points of intersection are separated by distance $2r$. Show that the area inside the small circle and outside the large circle is largest when $r = (1 + (2/\pi)^2)^{-1/2}$. Note: you don't need to know the anti-derivative of $\sqrt{1 - x^2}$.

(b) [**3 marks**] A sphere of radius 1 meter overlaps a smaller sphere of radius r in such a way that their intersection is a circle of radius r, i.e. they intersect in a great circle of the smaller sphere. Show that the volume inside the small sphere and outside the large sphere is given by:

$$V(r) = \frac{\pi}{3}[2r^3 - 2 + (r^2 + 2)\sqrt{1 - r^2}]$$

(c) [**3 marks**] For the same problem as in (b) find the value of r for which this volume is (i) largest, (ii) smallest.

10. [**10 marks**] You have to travel from one corner of a square lake, side α km, to the diagonally opposite corner in the shortest possible time. You can walk at β km/hr and swim at γ km/hr. Prove that, regardless of the values of α, β, γ that the shortest time always involves walking only or swimming only, never a combination. You may consider two possibilities involving a combination of walking and swimming: (i) walk distance a, then swim distance b, then walk distance a, (ii) walk distance c then swim distance d.