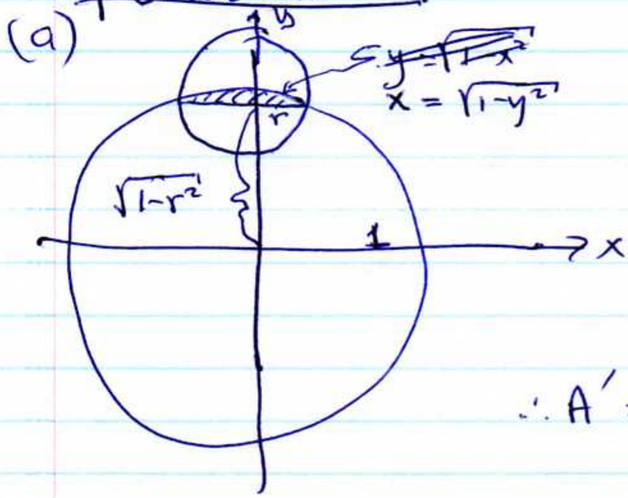
Question 9

(a)



Shaded area is A_1 $\quad \downarrow \frac{dA_1}{dr}$

$$A_1 = 2\int_{y=\sqrt{1-r^2}}^{y=1} \sqrt{1-y^2}\,dy \rightarrow A_1' = -\frac{2r^2}{\sqrt{1-r^2}}$$

area we want (inside small circle and outside large circle) is A

where $A = \frac{1}{2}\pi r^2 - A_1$

$\therefore A' = \pi r - \frac{2r^2}{\sqrt{1-r^2}}$ set $=0$ & solve for

$$r_m = \frac{1}{\sqrt{1+(2/\pi)^2}}$$

(b) Similar figure but spherical $\quad V = \frac{1}{2}\frac{4}{3}\pi r^3 - V_{cap}$

$$V_{cap} = \int_{y=\sqrt{1-r^2}}^{y=1} \pi(1-y^2)\,dy$$

$$= \pi\left(\frac{2}{3} - (1-r^2)^{1/2} + \frac{1}{3}(1-r^2)^{3/2}\right)$$

$$\therefore V(r) = \frac{\pi}{3}\left(2r^3 - 2 + 3(1-r^2)^{1/2} - (1-r^2)^{3/2}\right)$$

$$= \pi/3\left[2r^3 - 2 + (r^2+2)\sqrt{1-r^2}\right]$$

(c) $V' = \frac{\pi}{3}\left(6r^2 + 2r(1-r^2)^{1/2} - \frac{(r^2+2)r}{(1-r^2)^{1/2}}\right) \overset{\text{set}}{=} 0$ and solve for r_m

$\rightarrow r_m = 2/\sqrt{5}$

$V(0)=V(1)=0$ and since $V(r)$ is continuous, by EVT it does have a maximum on $r\in[0,1]$ $\therefore$ $V(r_m)$ can't be min, so must be max

V_{min} for $r=0$ or 1

Question 7

$y=0 \rightarrow x=\pm 1 \qquad \frac{d}{dx}(x^4 - xy + y^4 = 1)$

$\rightarrow 4x^3 - y - xy' + 4y^3y' = 0$

at $y=0$: $4x^3 - xy' = 0 \rightarrow y' = 4x^2 = 4$ for $x=\pm 1$

Question 2

$$\text{avg} = \int_0^6 \frac{5\,dx}{\sqrt{2x+1}} \Big/ 6 = 5/6\,(13^{1/2} - 1)$$

Question 5

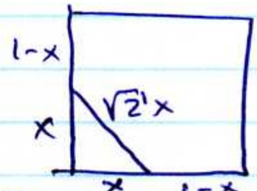
5a $\quad f'(0) = \lim_{\Delta\to 0} \frac{f(\Delta)-f(0)}{\Delta} = \lim_{\Delta\to 0} \Delta \sin\frac{1}{\Delta}$

Note: $-1 \le \sin\frac{1}{\Delta} \le 1 \quad \therefore -\Delta \le \Delta\sin\frac{1}{\Delta} \le \Delta$. Apply Sandwich Thm and $\lim_{\Delta\to 0}(\pm\Delta) = 0 \rightarrow \lim_{\Delta\to 0} \Delta\sin\frac{1}{\Delta} = 0$, exists

5b $\quad f'(0) = \lim_{\Delta\to 0} \sin\frac{1}{\Delta^2}$ DNE. Proof by contradiction. Assume limit exists $= a \in (-1,+1)$. Say $\varepsilon = (1-a)/2 > 0$. No matter what $\delta > 0$ is chosen there are some $-\delta < x < \delta$ s.t. $\sin\frac{1}{\Delta^2} = 1 > a+\varepsilon$ $\therefore$ limit DNE

Question 10

(a) First combination



without loss of generality we can make $\alpha = 1$, $\beta = 1$ (in effect changes units of time and distance)

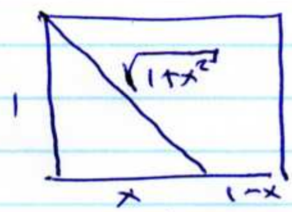
time required $T = 2(1-x) + \frac{\sqrt{2}x}{\gamma}$

$$= 2 + x\left(\frac{\sqrt{2}}{\gamma} - 2\right)$$

$\therefore$ if $\gamma < \frac{1}{\sqrt{2}}$, $\left(\frac{\sqrt{2}}{\gamma} - 2\right) > 0$ $\therefore T(x)$ increasing fcn $\therefore$ pick $x=0$ $\therefore T = 2$

if $\gamma > \frac{1}{\sqrt{2}}$, $\left(\frac{\sqrt{2}}{\gamma} - 2\right) < 0$ $\therefore T(x)$ dec. fcn. $\therefore$ pick $x=1$, $\therefore T = \sqrt{2}/\gamma$

(b) Second combination



$T = 1-x + \frac{\sqrt{1+x^2}}{\gamma}$

$\therefore T' = -1 + \frac{x}{\gamma\sqrt{1+x^2}}$

set $T'=0$ and solve to find $x_m = \gamma/\sqrt{1-\gamma^2}$, $T_m = 1 + \frac{\sqrt{1-\gamma^2}}{\gamma}$

$\therefore$ if $\gamma < \frac{1}{\sqrt{2}}$ $\rightarrow$ gives $T_m > 2$ $\therefore$ shorter to walk whole way

if $\gamma > \frac{1}{\sqrt{2}}$ $\rightarrow$ gives $x_m > 1$ which occurs outside the permitted domain, $x \in [0,1]$

$\therefore$ look for end-point min: $T'(1) < 0$ for $x > \frac{1}{\sqrt{2}}$ $\therefore$ end-pt min

($T'(0) = -1 < 0$ $\therefore$ end-pt max)

So $x=1$ is shortest i.e. swim whole way

Question 6

$f(x) = x^{2/5}(x - 7/2)$. Intercepts @ $x = 0, 7/2$ $\lim_{x\to\pm\infty} f = \pm\infty$

$f' = \frac{7}{5}x^{-3/5}(x-1)$. For $x<0$, $f'>0$ $\therefore$ f inc. For $0<x<1$, $f'<0$, f dec. For $x>1$ f inc.

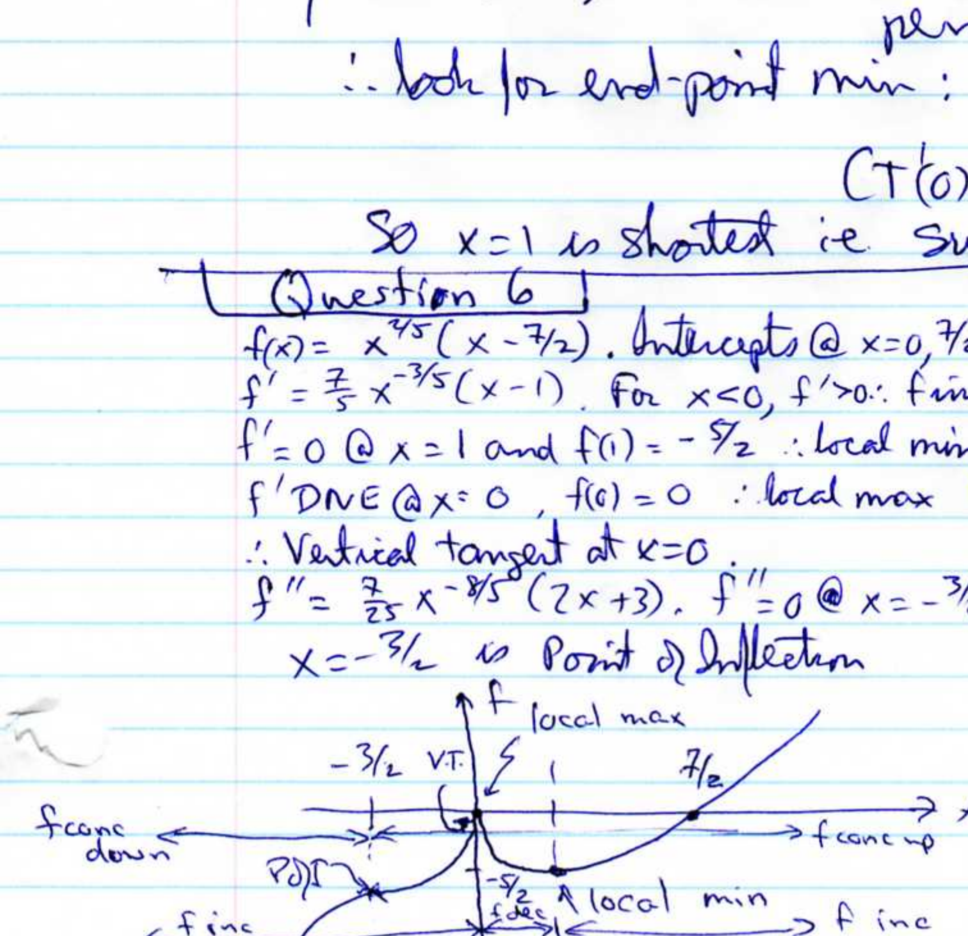
$f' = 0$ @ $x = 1$ and $f(1) = -5/2$ $\therefore$ local min (FDT)

f' DNE @ $x=0$, $f(0) = 0$ $\therefore$ local max (FDT) $\lim_{x\to 0^+} f' = -\infty$, $\lim_{x\to 0^-} f' = +\infty$

$\therefore$ Vertical tangent at $x=0$.

$f'' = \frac{7}{25}x^{-8/5}(2x+3)$. $f'' = 0$ @ $x = -3/2$. $f''<0$ for $x < -3/2$ $\therefore$ f conc d.

$x = -3/2$ is Point of Inflection $f''>0$ for $x > -\frac{3}{2}$ $\therefore$ f concu



MT 2 MAT194 Fall 2008

① We are given $x_1 < x_2 < x_3$ with $f(x_1) = f(x_2) = f(x_3) = 0$. By the mean value theorem, there exists $x_{12}^* \in (x_1, x_2)$ with

$$f(x_2) - f(x_1) = f'(x_{12}^*)(x_2 - x_1).$$

We conclude $f'(x_{12}^*) = 0$. Similarly, there exists $x_{23}^* \in (x_2, x_3)$ with $f'(x_{23}^*) = 0$. By the mean value theorem again, applied to f', there exists $y \in (x_{12}^*, x_{23}^*)$ such that $f'(x_{23}^*) - f'(x_{12}^*) = f''(y)(x_{23}^* - x_{12}^*)$. We conclude $f''(y) = 0$ so f'' has at least one root.

③ For f continuous, we are to show that

$$\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx.$$

- <u>First Solution:</u>

$$\int_0^1 f(1-x)\,dx \underset{\substack{u = 1-x \\ du = -dx}}{=} \int_1^0 f(u)(-du) = \int_0^1 f(z)\,dz = \int_0^1 f(x)\,dx.$$

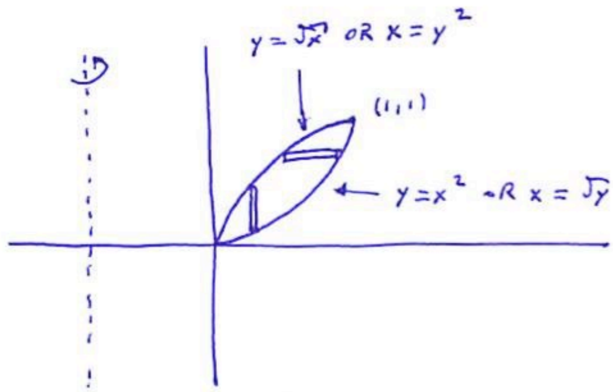
- <u>Second Solution</u>

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j^*)\,\Delta x \qquad \text{(Riemann Integral)}$$

$$\int_0^1 f(1-x)\,dx = \lim_{n\to\infty} \sum_{j=1}^{n} f(x^*_{(n+1)-j})\,\Delta x.$$

These limits are equal since the sums $\sum_{j=1}^{n}$ are rearrangements of each other.

④



SHELLS

$$\int_0^1 2\pi(1+x)(\sqrt{x}-x^2)\,dx = \int_0^1 2\pi\left(\sqrt{x}-x^2+x^{3/2}-x^3\right)dx$$

$$= 2\pi\left[\frac{2}{3}-\frac{1}{3}+\frac{2}{5}-\frac{1}{4}\right]$$

$$= 2\pi\left[\frac{1}{3}+\frac{3}{20}\right]$$

WASHERS

$$\int_0^1\left[\pi(1+\sqrt{y})^2-\pi(1+y^2)^2\right]dy = \pi\int_0^1\left(\cancel{1}+2\sqrt{y}+y-\cancel{1}-2y^2-y^4\right)dy$$

$$= \pi\left(\frac{4}{3}+\frac{1}{2}-\frac{2}{3}-\frac{1}{5}\right)$$

$$= \pi\left(\frac{2}{3}+\frac{1}{2}-\frac{1}{5}\right)$$

$$= 2\pi\left(\frac{1}{3}+\frac{1}{4}-\frac{1}{10}\right) = 2\pi\left(\frac{1}{3}+\frac{3}{20}\right).$$

⑤ $$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}.$$

(a)

$$f'(0) = \lim_{h\to 0}\frac{h^2\sin\frac{1}{h}}{h} = \lim_{h\to 0} h\sin\frac{1}{h}.$$

By the squeeze theorem, since $-h \le h\sin\frac{1}{h} \le h$, we see that $\lim_{h\to 0} h\sin\frac{1}{h} = 0$ so $f'(0)=0$ and exists.

(b) $f'(0) = \lim_{h\to 0}\sin\frac{1}{h}$ does not exist.

⑧ By the mean value theorem, for any points $a < b$ there exists a point $c \in (a,b)$ such that

$$|\cos a - \cos b| \leq |\sin(c)|\,|a-b|.$$

Since $|\sin(c)| \leq 1$ no matter what value of c, we have

$$|\cos a - \cos b| \leq |a-b|.$$