Heat and Mass Transfer

General Balance: same for all transfers: $\frac{dB_{CV}}{dt} = \dot{B}_{in} - \dot{B}_{out} + \dot{B}_{gen} - \dot{B}_{des}$ or the rate of change of B in the **control volume** (CV) is equal to amount of B input/output to/from the CV plus/minus any B generated/destroyed inside.
Control Volume: the volume which you consider as the system. You must always define this or you get zero.

Mass Balance: since mass cannot be created/destroyed we have: $\frac{dm_{CV}}{dt} = \dot{m}_{in} - \dot{m}_{out}$

Steady State: does not change or vary with time. In this case $d/dt = 0$. Ex. For steady state flow through a pipe of area A_1 to area A_2 the velocity changes, $v_2 < v_1$ for $A_2 > A_1$. Assuming density, ρ to be constant then:

$$0 = \rho v_1 A_1 - \rho v_2 A_2 \rightarrow v_2 = v_1 A_1 / A_2 .$$

Continuity Equation: derived in class using a volume of $\Delta x \Delta y \Delta z$ with in/out flux in each direction as for ex., $(\rho v_x)\Delta y \Delta z - (\rho v_x)\Delta y \Delta z$. Taking the limit of each of these and rearranging gave the continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0 \quad \text{or simply,} \quad \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho \vec{v}) = 0$$

For incompressible fluid, ρ doesn't vary in space or time: $\frac{\partial(v_x)}{\partial x} + \frac{\partial(v_y)}{\partial y} + \frac{\partial(v_z)}{\partial z} = \vec{\nabla}\cdot\vec{v} = 0$

For steady state, ρ can still vary in space: $\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = \vec{\nabla}\cdot(\rho\vec{v}) = 0$

Energy: the sum of kinetic, gravitational, and internal energy respectively is: $E = m\left(v^2/2 + gh + \hat{u}\right)$.

Energy Balance: since energy cannot be created/destroyed we have: $\frac{dE_{CV}}{dt} = \dot{E}_{in} - \dot{E}_{out}$

Energy In/Out Mechanisms:

1. Work: either by pressure/volume (*PdV*), shaft work, or electrical work.
2. Heat: Conductive heating is when particles collide with each other and transfer heat. Convective heating is between a solid/fluid interface. Resistive heating is heating within a solid.
3. Mass Transferring In/Out: can bring energy in with it: $\dot{m}_{in/out}\left(v^2/2 + gh + \hat{u}\right)$.
4. Conversion of ordered forms of energy into internal energy.

Fourier's Law of Heat Conduction: caused by the collision of particles. The heat flux, q, is defined by:

$$q_x = -k\frac{dT}{dx} \quad \text{(x-direction)} \quad \text{or} \quad \vec{q} = -k\nabla T \quad \text{(in general)}$$

Where k is the **thermal conductivity constant** and T is the temperature. The units of q are $J/\left(m^2\cdot s\right)$. To get the total heat flux for a given area, $\dot{Q}$, you must multiply q by the area, A. Notice the heat flux points down the temp. gradient, thus a decrease in temperature represents a positive heat flux.
General Method of Solving Balance Questions: for ex., a heat transfer through a slab of width L and area A:

- Define CV as a thin cut out slab, $x \rightarrow x + \Delta x$ from the larger slab, still with cross sectional area, A.
- Write out the balance equation, assuming steady state here: $0 = (q_x)_x A - (q_x)_{x+\Delta x} A$.
- Divide by differential value, Δx and take limit, which is the derivative: $\lim_{\Delta x \to 0}\frac{(q_x'')_{x+\Delta x} - (q_x'')_x}{\Delta x} = \frac{dq_x''}{dx} = 0$.
- Integrate this to get $q_x'' = c_1$ and set equal to the corresponding law: $q_x'' = c_1 = -kdT/dx$.
- Rearrange and integrate both sides to get: $\int dT = \int -(c_1/k)dx \rightarrow T = -(c_1 x/k) + c_2$

- Apply **Boundary Conditions** to solve for the constants. Here, $@x=0,T=T_0$ and $@x=L,T=T_L$.
- Write out the relationship with the constants in place: $\frac{T(x)-T_0}{T_L-T_0}=\frac{x}{L}$. This is the relationship required.
- Aside: Rearranging for $T(x)$ and plugging into Fourier's Law gives: $q_x''=k(T_0-T_L)/L$.

Newton's Law of Cooling: describes the effects of convection. The heat flux in this case is: $q_0''=h(T_0-T_\infty)$ This is the heat flux from the surface at T_0 to the bulk fluid next to it at T_∞. In general it is in the direction from the first temp to the second temp written. h is the **convective heat transfer coefficient.** The sign of (T_0-T_∞) determines if heating or cooling takes place. Ex. Two slabs in contact with each other with fluid at T_{in} touching the T_1 slab and T_{out} fluid touching the T_2 slab. First write out each heat flux:

$$q_1''=\frac{k_1(T_0-T_1)A}{L_1},\quad q_2''=\frac{k_2(T_1-T_2)A}{L_2},\quad q_{in}''=h_{in}(T_{in}-T_0)A,\quad q_{out}''=h_{out}(T_2-T_{out})A$$

At steady state: $q''=q_1''=q_2''=q_{in}''=q_{out}''$. Rearranging all for their (T_i-T_j) terms and adding to cancel temps. gives:

$$\Delta T_{overall}=T_{in}-T_{out}=q\left[\frac{1}{h_{in}A}+\frac{1}{Ak_1/L_1}+\frac{1}{Ak_2/L_2}+\frac{1}{h_{out}A}\right]$$

This can be rearranged for q. This is similar to Ohm's Law with $T=V$, $q=I$, and $R_{conv.}=\frac{1}{h_iA_i}$, $R_{cond.}=\frac{1}{A_ik_i/L_i}$.

These resistances can even be added in parallel when slabs are next to each other, just like parallel resistors.
Resistive Heating: denoted Se in units $J/m^3\cdot s$, thus you must multiply by the volume before you place this term on the right side of the energy balance equation. Ex. For a cylindrical problem follow the steps as above and set q equal to Fourier's Law at the curved surface of the cylinder. Using the boundary conditions,

$@r=R,T=T_R$ and $@r=0,const=0$ (otherwise q would go to infinity). $\therefore T(r)=\frac{-SeR^2}{4k}\left[1-\left(\frac{r}{R}\right)^2\right]+T_R$.

Total Heat Loss: from the cylinder example above it is: $\dot{Q}=(q_r)_R(A)_R=\frac{SeR}{2}\cdot 2\pi RL=Se\pi R^2L$. This can also be found if the CV is the entire cylinder: $0=SeV-\dot{Q}_{loss\ surface}=Se\pi R^2L-\dot{Q}_{loss\ surface}$.

Predicting Differential Direction: if you are given many variables in which your CV could be differential with respect to such as: $T(r,\theta,z,t)$ you have to chose one to focus your problem around. If steady state, time is irrelevant. For things such as a cylinder, the angle is irrelevant since at each angle around it should be the same. So you are left with z and r, which you have to determine which has a bigger effect in the problem. You do this by relating them to their derivatives, $\partial T/\partial z \ll \partial T/\partial r$, since $z\gg r$ for a cylinder, thus you would chose $T(r)$.

Species Mass Balance: the transfer of particles diffusing through a fluid. The main unit is N_A which is the moles of species A. Also, the mass of A, m_A can be used. The balanced equation is:

$$\frac{dN_A}{dt}=\dot{N}_{A,in}-\dot{N}_{A,out}+\dot{N}_{A,gen}-\dot{N}_{A,dest}$$

The generations and destruction can occur through chemical reactions in this case.
1-Compartment Model: consider a control volume V, that has an injection at $t=0$. The **concentration** is given by $N_A=VC_A$. A rate constant k is used to represent how fast A is cleared or destroyed from the volume. Thus:

$\frac{d[VC_A(t)]}{dt}=-kC_A(t)$ integrating with BC→ $t=0,C_A=C_{A0}$ gives: $C_A(t)/C_{A0}=\exp[-kt/V]$

2-Compartment Model: there are now two control volumes V_1 and V_2 with a transfer rate constants: k_1 between them, k_2 out of CV2, and k_c being cleared from CV1. Moles of A are introduced to CV1 at a rate of $\dot{N}_A$. BC's: $@t=0,C_A=0\ and\ \underline{C}_A=0$ ($\underline{C}_A$ is in CV2).

$$CV1:\frac{d[V_1C_A(t)]}{dt}=\dot{N}_{A,in}-k_cC_A-k_1C_A \quad\text{and}\quad CV2:\frac{d[V_2\underline{C}_A(t)]}{dt}=k_1C_A-k_2\underline{C}_A$$

Species In/Out Mechanisms:

1. Carried in/out by flowing fluids.
2. **Molecular Diffusion:** (like heat conduction) from high to low concentrations: $J_{Ax}=-DC\frac{dy_A}{dx}$
3. **Convective Mass Transfer:** (like heat convection) which occurs when a reaction takes place at a surface or a substance can go through the solid wall material. $(N_{Ax})_{surface}=+k_c(C_{As}-C_{A\infty})=-D\frac{dC_A}{dx}$ where C_{As} is the concentration at the surface, $C_{A\infty}$ is the bulk fluid concentration, and k_c is the convective rate const. This type of transfer always goes from higher to lower concentrations.

Types of Species Flux: $\vec{N}_A$ is the absolute flux (how A flows) and $\vec{J}_A$ is the diffusive flux (how A diffuses).
Fick's Law of Diffusion: defines the diffusive flux $J_{Ax}=-DC\frac{dy_A}{dx}$ if $C=const$ then $J_{Ax}=-D\frac{dC_A}{dx}$. y_A is the mole fraction of A to the sum of all moles in solution.
Absolute Flux: is $N_{i,x}=v_{i,x}C_i$ in $mols/m^2\cdot s$ where $v_{i,x}$ is the absolute x velocity of species i and $C_i=N_i/V$.
Molar Average Velocity of Mixture: $v_x=\left(\sum_{i=1}^{N}N_iv_{i,x}\right)/\left(\sum_{i=1}^{N}N_i\right)=\left(\sum c_iv_{i,x}\right)/\left(\sum c_i\right)=\left(\sum N_{i,x}\right)/C$ which is the total absolute flux divided by the total concentration.
Relationship Between Absolute and Diffusive Fluxes: The diffusive flux can be written as: $J_A=C_A(v_{Ax}-v_x)=C_Av_{Ax}-C_Av_x=N_{Ax}-C_Av_x$ where $(v_{Ax}-v_x)$ is the velocity of A relative to the molar average velocity. From above: $N_{Ax}=J_{Ax}+C_Av_x=J_{Ax}+C_A\left(\sum N_{\alpha x}\right)/C=J_{Ax}+y_A\sum N_{\alpha x}$. This states that the absolute flux is the sum of the diffusive flux and the bulk motion. In general for two species in solution: $N_A=-CD_{AB}\nabla y_A+y_A(N_A+N_B)$ which becomes: $N_A=-D_{AB}\nabla C_A=J_A$ when the total concentration C is constant and the solution is dilute wrt. A ($y_A\rightarrow 0$).

Various Boundary Conditions for Species Mass Transfer: 1) If the concentration is known at $z=0,C_A=C_{A0}$. 2) Symmetry about a center line of the CV for diffusion or no net diffusive flux at an interface, $N_A=0=dC_A/dz$. 3) A convective flux at a surface, $N_A=k_c(C_{As}-C_{A\infty})$. 4) A known flux at an interface, $N_A|_{z=0}=N_{A0}$. 5) Rapid reaction disappearance at surface for species A, $@z=0,C_{As}=0$. 6) For a slower reaction with first order rate constant k' then, $@z=0,N_A=k'C_{As}$.

Sherwin Number, Sh: relates the convective mass transfer coefficient, k_c, to other quantities. The Sherwin Number is the unitless whole number always. For a wall: $\frac{k_c}{D/L}$, for a sphere radius R: $\frac{k_c}{D/R}=1$. k_c relates to the reaction speed at a surface and D relates to the diffusion speed through the fluid touching the surface.

Diffusion Limitations: for a flat surface at $x=0$ where a reaction, $R_A = -kC_{A0}$ occurs at the surface, this reaction must equal $D(C_{AL} - C_{A0})/L$ at steady state otherwise accumulation would occur.

Setting these equal and solving gives: $C_{A0} = \dfrac{C_{AL}}{1+(kL/D)}$. As $\dfrac{k}{D/L} \to 0, C_{A0} \to C_{AL}$ and as $\dfrac{k}{D/L} \to \infty, C_{A0} \to 0$.

Effectiveness Factor, η**:** the actual rate compared to the maximum possible rate. The actual rate is often integrated over all control volumes while the max is often the flux times the entire area:

$$\eta = \frac{\text{actual rate}}{\text{max possible rate}} = \frac{kC_{A0}\cdot S}{kC_{AL}\cdot S} = \frac{C_{A0}}{C_{AL}} = \frac{1}{1+(kL/D)} \quad \text{(for a wall of surface area } S\text{)}$$

Diffusive Pore: this is a cylinder protruding from a wall. There are three possible boundary conditions:

1. Long Pore Assumption ($x \to \infty$): $@x = L, C_A = 0$.
2. Inert End Assumption, no diffusion between in/out concentrations: $@x = L, dC_A/dx = 0$ (flux = 0).
3. Assume End is Catalytic, so reaction occurs there: $@x = L, -DdC_A/dx = kC_A \neq 0$. (Conduc. = Convect.)

There are analogous BC's for heat transfer down a heat fin as well.

Interphase Mass Transfer: this involves the diffusion from fluid to solid and vice versa just like for the heat transfer case. You must convert all the concentrations to a convenient reference concentration (denoted with an overbar). In a fluid to solid to fluid mass transfer, the middle solid concentration, $\bar{C}_{A,2}$ would be chosen as an example. This conversion is done by finding constants K_1 and K_3 which relate the concentrations in area 1 and 3 to the concentration in area 2 at equilibrium. Write: $C_{A,1} = K_1\bar{C}_{A,2}$ and $C_{A,3} = K_3\bar{C}_{A,2}$ where $K_1 = m^3 in 2 / m^3 in 1$ and $K_3 = m^3 in 2/m^3 in 3$. If $C_{A,m}$ is in fluid 1, $C_{A,1}$ is at the boundary at fluid 1, $\bar{C}_{A,1}$ is on the other side of the boundary in the solid and so on, then you can write:

Fluid 1: $N_{Az} = k_{c,1}(C_{A,m} - C_{A,1}) = k_{c,1}(K_1\bar{C}_{A,m} - K_1\bar{C}_{A,1})$ Solid: $N_{Az} = D(\bar{C}_{A,1} - \bar{C}_{A,2})/L$

Fluid 2: $N_{Az} = k_{c,3}(C_{A,2} - C_{A,3}) = k_{c,3}(K_3\bar{C}_{A,2} - K_3\bar{C}_{A,3})$

Rearranging, cancelling concentrations, and then solving for N_{Az}: $\bar{C}_{A,in} - \bar{C}_{A,out} = N_{Az}\left[\dfrac{1}{K_1k_{c,1}} + \dfrac{1}{D_2/L_2} + \dfrac{1}{K_3k_{c,3}}\right]$.

This is analogous to the heat transfer with $R_{convective} = \dfrac{1}{K_ik_{c,i}}$ and $R_{diffusive} = \dfrac{1}{D_i/L_i}$.

Dissolution Rate: is the transfer of mass per second through an area, S, given by: $W_A = S\cdot N_{Az}$.

Fluid Mechanics

Fluid: a substance which deforms continuously under the application of a shear stress of any magnitude.

Density: $\rho(\bar{r}) = \lim_{\delta V\to 0} \delta m/\delta V$

Continuum Hypothesis: the smallest part of fluid has the same properties as the bulk material.

Body Force: any force acting on a particle proportional to δm. For gravity, $\delta\bar{F}_b = \delta m\bar{g} = \rho\bar{g}\delta V$.

Stress Field: normal forces act perpendicular to surfaces while tangential forces act parallel to surfaces.

Viscosity: a measure of the resistance of a fluid to shear stress. If a fluid is contained between two plates, the bottom one being stationary and the top moving at speed U then the **shear stress** is : $\tau = u\dfrac{du}{dy}$ where u is the velocity and y is the up direction normal to the bottom plate. The top fluid moves at U and the bottom at 0.

Bulk Modulus: measure of resistance to compression. Defined as $E_v = \dfrac{-dP}{dV/V} = \dfrac{-dP}{d\rho/\rho}$. Large E_v means incomp.

Hydrostatics

We are only concerned with normal forces since by definition shear forces produce movement.

Pressure Force: of fluid on a surface, S, depends on pressure P and acts normal to the surface ($\hat{n}$). The force is:

$$\delta F_p = -P\hat{n}dS \to \delta F_p = -\nabla P(\bar{r})dV \quad \text{(from Gradient Theorem)}$$

Pressure at a point acts equally in all directions.

Pressure Axioms: 1) pressure acts along $\hat{n}$, 2) is proportional to dS, and 3) is compressive (hence the neg.).

Equations of Hydrostatics: equating the pressure and body forces gives:

$$-\nabla PdV = -\rho gdV \to \nabla P = \rho\bar{g} \quad \text{or} \quad \frac{\delta P}{\delta x}\hat{i} + \frac{\delta P}{\delta y}\hat{j} + \frac{\delta P}{\delta z}\hat{k} = \rho g_x\hat{i} + \rho g_y\hat{j} + \rho g_z\hat{k}$$

Another proof in equilibrium: $F_b + F_p = 0 = \int_V \rho\bar{g}dV - \int_S P\hat{n}dS = \int_V \rho\bar{g} - \nabla PdV = 0$, V is arbitrary so $\to \nabla P = \rho\bar{g}$

Pressure at a given depth, h, is typically: $P = \rho gh$.

Flow From a Reservoir: $\frac{1}{2}mv^2 = mgh \to v = \sqrt{2gh}$. A flow, Q, through an area, A, is $Q = Av = A\sqrt{2gh}$.

Pressure Gauge: reads the difference in pressure so the actual pressure is ($P_{reading} + 1\,atm$). This 1 atm usually cancels out in calculations since other surfaces typically have this pressure exerted on them.

Coordinate System: always set up a coordinate system to make calculations easiest. Whenever there is a hinge involved, always setup the origin of the coordinate system at the hinge.

Archimedes Principle: the buoyant force on an object floating in fluid is equal to the force of gravity on the amount of fluid displaced by the object. The buoyant force acts upwards on the center of mass of the displaced volume of fluid while the gravitational force on the object acts at the center of mass of the object.

Buoyancy/Stability: when the gravitational force is acting too far outside the buoyant force that there is no chance of restoring to its original location thus this is an unstable moment. Other cases are stable moments.

An Accelerated Box: a non-inertial force, $\bar{F}_a$ caused by an acceleration of the reference frame is introduced:

$$\delta\bar{F}_p + \delta\bar{F}_b + \delta\bar{F}_a = 0 = -\nabla P\delta V + \rho\bar{g}\delta V - \bar{a}\delta m \to \frac{-\nabla P}{\rho} + \bar{g} = \bar{a}$$

Steps to Solving Pressure Problems:

1. Choose coordinate system.
2. List variables such as acceleration and their directions, $(\hat{i},\hat{j},\hat{k})$ or $(\hat{r},\hat{\theta},\hat{\phi})$.
3. Write $\dfrac{-\nabla P}{\rho} + \bar{g} = \bar{a}$ split into each of the three direction components.
4. Solve each component of P through integration.
5. Add up all the P components. Add all constant functions from integration together giving a constant, C.
6. Apply boundary conditions, for example $@z = z_S$, $P = P_{atm}$ and assume constant water volume. If the volume is constant then $(L\cdot W\cdot H)_{original} = \int_0^W\int_0^L z_S dxdy$, where z_S is the surface. Solving for z_S gives a function of the fluid's surface when under the given movements.

Dimensional Analysis

Let M, L, and T be the **fundamental dimensions** representing mass, length, and time. Common units are thus:

$v: LT^{-1}$ velocity	$\rho: ML^{-3}$ density	$g: LT^{-2}$ acceleration	$\mu: MT^{-1}L^{-1}$ viscosity
$\omega: T^{-1}$ angular frequency	$P: ML^{-1}T^{-2}$ pressure	$T, F: MLT^{-2}$ thrust, force	

Common Dimensionless Numbers: Reynolds #: $Re = \frac{PLv}{\mu} = \frac{\text{inertia}}{\text{viscosity}}$, Mach #: $Ma = \frac{v}{a}$ where a is speed of sound, Froude #: $Fr = \frac{v^2}{gL}$, and Strouhal #: $St = \frac{\omega L}{v} = \frac{\text{local change}}{\text{convective change}}$.

Pi Theorem: used to write variables as dimensionless values. Use these steps for the ex. $T = f(D, \rho, \mu, \omega, v)$:

1. Choose D, v, ρ as independent variables, (you always want at least one *M*, *L*, and *T* somewhere in the three independent variables). Note: # of variables - # of dimensions (indep. var.'s) = # pi groups
2. Place these into a matrix with the other variables and the overall variable with the powers of *M*, *L*, or *T*'s:

	D	v	ρ	μ	ω	T
M	0	0	1	1	0	1
L	1	1	-3	-1	0	1
T	0	-1	0	-1	-1	-2

3. Row reduce this until you have the identity matrix in the first three columns (the independent variables):

	D	v	ρ	μ	ω	T
M	1	0	0	1	-1	2
L	0	1	0	1	1	2
T	0	0	1	1	0	1

4. Write out the dependent variables in terms of the independent variables by looking at which independent variable corresponds to *M*, *L*, or *T* (where the 1 is) and then go down the dependent variable's column in which the numbers there are the powers of the independent variables corresponding to *M*, *L*, or *T*. Then write each as a pi group according to:
5. $\pi_{dep.var.} = \frac{\text{dep. var.}}{\text{indep. var's to powers given in the dep. var.'s col.}}$, $\pi_\mu = \frac{\mu}{\rho v D}$, $\pi_\omega = \frac{\omega D}{v}$, and $\pi_T = \frac{T}{\rho v^2 D^2}$.
6. Once all the pi groups are written state that: $\pi_T = f(\pi_\omega, \pi_\mu)$.

Fluid Dynamics

Eulerian Approach: use $\rho = \rho(\vec{r}, t)$, $\vec{v} = \vec{v}(\vec{r}, t)$, and $\vec{v}(\vec{r}, t) = u\hat{i} + v\hat{j} + w\hat{k}$.

Fluid Speed: denoted $q = \|\vec{v}\| = \sqrt{u^2 + v^2 + w^2}$.

Steady Flow: does not depend on time, $\vec{v} = \vec{v}(\vec{r})$.

Unsteady Flow: does depend on time, $\vec{v} = \vec{v}(\vec{r}, t)$.

Incompressible Flow: $\rho_{particle}$ = constant. You can have two fluids together, but each density doesn't change.

Constant Density Flow: ρ = same for all particles in the flow.

Streamline: a line drawn everywhere tangent to $\vec{v} = \vec{v}(\vec{r}, t)$.

Streamtube: (steady flow) a closed curve *C* enclosing a surface *S* forms a streamtube out of the streamlines that pass through it. All the fluid that passes through *S* stays confined to the shape of *S* since fluid can't by definition pass through a streamline (treat a streamtube as if it is a physical tube).

Cross Section of Streamtube: a cross section has an area of A_p whose normal is: $\hat{n} = \frac{\vec{v}_p}{\|\vec{v}_p\|}$.

Stream Filament: a small stream tube with cross section $A_f \to \delta A_f \to a_f$. $\vec{v}$ is constant over a_f. Let *s* be the direction down a stream filament, then the shape is denoted: $\vec{r}_p = r_p(s)$, the cross section is: $a_f = a_f(s)$, density is: $\rho = \rho(s)$, and velocity is $\vec{v}(\vec{r}) = \vec{v}(s) = q(s)\hat{i}_s$.

Equations of Motion: since mass cannot be created nor destroyed, the volume flow rate is:

$$a_1 q_1 = a_2 q_2 = Q \quad \text{(incompressible, steady flow)}$$

Applying $F = ma \to \partial\vec{F}_b + \partial\vec{F}_p = \delta m\vec{a}$ gives:

$$-\frac{\nabla P}{\rho} + \vec{g} = \vec{a} \quad \text{(same as for non-inertial reference frame)}$$

Note: Steady flow can have acceleration since you can move from regions of fast flow or slow flow.

Euler's Equation: for steady, inviscid (having no viscosity), compressible flow:

$$q\frac{dq}{ds} + g\frac{dz}{ds} + \frac{1}{\rho}\frac{dP}{ds} = 0 \quad \text{where } a_s = q\frac{dq}{ds} \text{ is the convective acceleration}$$

You cannot integrate this unless $\rho = const$ or you know the ρ function.

Bernoulli's Law: assuming $\rho = const$ and integrating the above for steady, inviscid flow:

$$\frac{P}{\rho} + gh + \frac{1}{2}q^2 = const = E_T$$

Where E_T is **Bernoulli's constant** which stays the same at all parts of a flow, thus can be used to solve for various variables by comparing flow at two different locations.

Momentum Balance on a Filament: $\frac{d\vec{p}}{dt} = \vec{F}_E$ = resultant of all external forces. This can be re-written in terms of **mass flux**, $\dot{m}_f = \rho a_f q = const$ since mass cannot be create/destroyed. Thus: $\frac{d\vec{p}}{dt} = \vec{F}_E = \dot{m}_f(\vec{v}_2 - \vec{v}_1)$.

Moving Reference Frame: assume a nozzle shoots water at $\vec{v}_1 = q_1\hat{i}$, towards a vane that moves at $\vec{v}_0 = u_0\hat{i}$. The velocity in the moving reference frame would then be: $\vec{w}_1 = \vec{v}_1 - \vec{v}_0 = (q_1 - u_0)\hat{i}$.

General Form of Conservation of Mass: assuming two locations in a flow, each with $a_i \perp \vec{v}_i$ and $\hat{n}_i \perp \delta S_i$. From the continuity equation for incompressible flow, $a_1 q_1 = a_2 q_2 = Q$ we get:

$$\vec{v}_1 \cdot \hat{n}_1 dS_1 + \vec{v}_2 \cdot \hat{n}_2 dS_2 = 0 \to \int_{S_1} \vec{v} \cdot \hat{n} dS + \int_{S_2} \vec{v} \cdot \hat{n} dS = 0$$

Integral Continuity $\int_{S_c} \vec{v} \cdot \hat{n} dS = 0$ **Differential Continuity** $\nabla \cdot \vec{v} = 0$ (incompressible)

Volume Flux Through a Surface: is the amount of fluid flowing through a surface:

$$Q = \lim_{dt \to 0} \frac{dV_S}{dt} = \int_S \vec{v} \cdot \hat{n} dS \quad \text{(steady or unsteady)}$$

Mass Flux Through a Surface: in this case the density doesn't have to be constant:

$$\dot{M} = \lim_{dt \to 0} \frac{dM_S}{dt} = \int_S \rho\vec{v} \cdot \hat{n} dS$$

General Form of Conservation of Mass (cont'd): the rate of increase of mass within a volume, V_c, must equal the negative of the mass flux out of the surface (since the integrals give the mass flux outwards):

$$\int_{V_c} \frac{d\rho}{dt} + \nabla \cdot (\rho\vec{v}) dV = 0$$

But since the volume is arbitrary we can write the conservation of mass as:

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\vec{v}) = 0 \quad \text{or} \quad \frac{D\rho}{Dt} + \rho\nabla \cdot \vec{v} = 0 \quad \text{(unsteady, compressible)}$$

Total Derivative: $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$ the first term is the local change, the others are convect. change.

Linearization Technique: used to convert complicated equations into a simpler form. This is done by changing all the variables to: $F = F + f$ where f is some function such that $f \ll F$. Place these new variables into each formula and expand out all terms. Cancel any terms that have multiples of the small functions since a square of a small number is an even smaller number, thus negligible.

One Dimensional Unsteady Flow: $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$ (continuity eq'n), $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0$ (momentum eq'n).

Wave Equation: $\frac{\partial^2 P}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2}$ where c is the speed of sound, $c^2 = \frac{dP}{d\rho}$.

Ideal Gas Law: $P = \rho RT$ which leads to: $c = \sqrt{RT}$. For **adiabatic** (no heat leaving), $P/\rho^\gamma = const, c = \sqrt{\gamma RT}$.

Bulk Modulus: $E = \rho\frac{dP}{d\rho} \rightarrow c = \sqrt{\frac{E}{\rho}}$ which is the speed in an incompressible fluid.

Compressible Conservation of Mass: for steady, inviscid, and adiabatic. Differentiate, $\rho_1\sigma_1 q_1 - \rho_2\sigma_2 q_2 = 0$:

$$\frac{1}{\rho}\frac{d\rho}{ds} + \frac{1}{\sigma}\frac{d\sigma}{ds} + \frac{1}{q}\frac{dq}{ds} = 0$$

Internal Energy: defined as: $e = (\text{internal energy})/(\text{unit mass})$.

Compressible Bernoulli Equation: for steady, inviscid, and adiabatic flow is:

$$\frac{P}{\rho} + e + \frac{1}{2}q^2 + gz = const \rightarrow h + \frac{1}{2}q^2 + gz = const \quad \text{where} \quad h = \frac{P}{\rho} + e$$

This does not work for viscous flow or unsteady flow.

Compressible Energy Equation: for steady, inviscid, and adiabatic flow. Differentiate the above and put into Euler's Equation:

$$\frac{de}{ds} - \frac{P}{\rho^2}\frac{d\rho}{ds} = 0 \quad \text{or} \quad \frac{1}{P}\frac{dP}{ds} - \frac{\gamma}{\rho}\frac{d\rho}{ds} = 0$$

Where the 2nd eq'n is found using: $e = C_V T,\ h = C_P T,\ \gamma = \frac{C_P}{C_V},\ c^2 = \frac{dP}{d\rho}, and\ P = \rho RT$. Integrating the above:

$$P = c_1\rho^\gamma,\ P = c_2 T^{\gamma/(\gamma-1)},\ and\ \rho = c_3 T^{1/(\gamma-1)}$$

Stagnation: the point at which there is no velocity. The variables here are denoted $\rho_0,\ P_0,\ and\ T_0$. To convert between stagnation and normal values use:

$$\frac{T_0}{T} = 1 + \left(\frac{\gamma-1}{2}\right)M^2,\quad \frac{\rho_0}{\rho} = \left(1 + \left(\frac{\gamma-1}{2}\right)M^2\right)^{1/(\gamma-1)},\ \text{and}\ \frac{P_0}{P} = \left(1 + \left(\frac{\gamma-1}{2}\right)M^2\right)^{\gamma/(\gamma-1)}$$

Note: when $M > 0.3$, compressibility must be taken into account.

Compressible 1-D Channel Flow: using continuity, momentum, and energy equations you get:

$$\frac{1}{A}\frac{dA}{ds} = -\left(1 - M^2\right)\frac{1}{q}\frac{dq}{ds} \ \text{(compressible)} \quad \text{or} \quad \frac{1}{A}\frac{dA}{ds} = -\frac{1}{q}\frac{dq}{ds} \ \text{(incompressible)}$$

If $M < 1$, $\frac{dA}{ds} < 0$ so increasing area decreases speed. If $M > 1$, $\frac{dA}{ds} > 0$ so increasing area increases speed.

2-D Incompressible Flow: $\vec{v} = u(x,y,t)\hat{i} + v(x,y,t)\hat{j}$ and $\vec{g} = g_x\hat{i} + g_y\hat{j}$ so:

$$\frac{d\rho}{dt} + \nabla\cdot(\rho\vec{v}) = 0 \rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \ \text{(Continuity Equation)}$$

$$-\frac{\nabla P}{\rho} + \vec{g} = \vec{a} \rightarrow \frac{D}{Dt}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] = \frac{D}{Dt}[\nabla\times\vec{v}] = 0 \ \textbf{(Helmholtz Theorem)}$$

Vorticity: $\nabla\times\vec{v}$. If $\nabla\times\vec{v} = 0$ the flow is called irrotational since each fluid element keeps its orientation.

Vector Potential: $\nabla\times\vec{v} = 0 \rightarrow \vec{v} = \nabla\phi$ where ϕ is called the vector potential.

Laplace's Equation: $\nabla^2\phi = 0$.

The Stream Function: for 2-D incompressible, inviscid flow. The stream function is denoted ψ and given by:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \rightarrow u = \frac{\partial\psi}{\partial y}\ and\ v = -\frac{\partial\psi}{\partial x} \ \text{since} \ \frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial x\partial y} = 0$$

A line where ψ is constant plots out a streamline for the flow, therefore:

$$d\psi = 0 = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = -vdx + udy \rightarrow \frac{dy}{dx} = \frac{v}{u}$$

If you choose a potential function as $\nabla\phi = \vec{v}$ where $\frac{d\phi}{dx} = u\ and\ \frac{d\phi}{dy} = v$ then ϕ and ψ are always perpendicular.

Bernoulli Theorem for 2-D Steady Flow: irrotational, steady, inviscid, incompressible:

$$\frac{P}{\rho} + \frac{1}{2}q^2 - g_x x - g_y y = const$$

No-slip Hypothesis: fluid adjacent to a solid surface does not move with respect to that surface.

Parallel Shear Flows: steady, incompressible, and viscid. As shear stress acts on a fluid element, it will shift from a square element to a parallelogram element by an angle θ. The shear stress is then given by:

$$\tau = \mu\frac{d\theta}{dt} \quad or \quad \tau = \mu\frac{du}{dy}$$

The direction of the shear stress is always denoted to the right of the normal direction on the top and bottom.

Solving Shear Stress Problems: follow these steps:

1. Draw the fluid particle.
2. Draw shear and pressure forces onto the particle. $F = \tau A\ or\ F = PA$ where the area's normal is either parallel to the pressure force or perpendicular to shear force.
3. Rearrange and cancel $\delta x, \delta y, \delta z$ terms by dividing through or multiply on the top and bottom and then letting $(P_1 - P_2)/\delta x = \partial P/\partial x$, etc.
4. Sub in $\tau = \mu\frac{du}{dy}$ and integrate.
5. Apply BC's such as no-slip, $Q = \int_S \vec{v}\cdot\hat{n}dS$, etc.

Linear Momentum Balance: rate of accum. of $m\vec{v}$ = rate of $m\vec{v}$ in – rate of $m\vec{v}$ out. In/Out mechanisms:

1. Mass in/out: $\dot{m}\vec{v}$
2. Forces on surface or body: $m\vec{v}/time$

By balancing all the surface forces and body forces to $d\left[\rho\Delta x\Delta y\Delta z v_x\right]/dt$ (this is for the x-direction). Dividing both sides by $\Delta x\Delta y\Delta z$, taking the limit as these approach zero, and assuming the fluid is incompressible gives:

Navier-Stokes Equation: assuming $\rho = const$, only gravitational forces, and Newtonian stress/strain:

$$\rho\frac{D\vec{v}}{Dt} = \mu\vec{\nabla}^2\vec{v} - \vec{\nabla}P + \rho\vec{g}$$

Energy Equation: if $\dot{q} = \frac{heat\ gen.}{vol.\cdot time}$ and $\Phi = \frac{visc.dissipation}{volume\cdot time}$ then similar to the Navier-Stokes equation:

$$\rho c_v\frac{DT}{Dt} = k\vec{\nabla}^2 T + \dot{q} + \Phi$$

Species Continuity: if $R_A = volumetric\ production\ rate$ then again similar to above: $\frac{DC_A}{Dt} = D\vec{\nabla}^2 C_A + R_A$.

Rearranging each of the above formulas with: $\nu = \frac{\mu}{\rho}$, $\alpha = \frac{k}{\rho c_P}$ and $c_V \sim c_P$ for solids and liquids:

$$\frac{D\bar{v}}{Dt} = \nu \bar{\nabla}^2 \bar{v} + \left[-\frac{\bar{\nabla} P}{\rho} + \bar{g}\right], \quad \frac{DT}{Dt} = \alpha \bar{\nabla}^2 T + \left[\frac{\dot{q} + \Phi}{\rho c_P}\right], \quad \frac{DC_A}{Dt} = D\bar{\nabla}^2 C_A + [R_A]$$

These forms are used to apply boundary conditions to.

General Relationships: at steady state all $d/dt = 0$.

For Temperatures:

- For temperature questions with no velocity: $v_x, v_y, v_z = 0$.
- If there is only a temperature gradient in the x-dir then: $d/dy, d/dz = 0$.
- If no generation or viscous dissipation then: $\dot{q}, \Phi = 0$.

For Flow in Pipes:

- Fully developed flow means: $d/dz = 0$.
- Expect v_z as a function of r.

Transient Flow/Conduction/Diffusion in a Semi-Infinite Medium

For the case where fluid next to a surface has, $v = v_S$, $T = T_S$, and $C_A = C_{AS}$ and $v_\infty, T_\infty, C_{A\infty}$ in the bulk fluid in which the surface moves up (y-direction) in the velocity case, or has a certain temp. or conc. for heat/mass transfers cases respectively. At various times t, the effects in the fluid are felt farther from the surface (in x-dir).

Dimensionalize Variables: the following are common variables used to dimensionalize equations:

$$\phi = \frac{v - v_0}{v_s - v_0}, \quad \theta = \frac{T - T_0}{T_s - T_0}, \text{ or } \psi = \frac{C_A - C_{A0}}{C_{As} - C_{A0}}$$

Similarity Solutions: a method for converting PDE's to ODE's so that you can solve them. Use the variables:

$$\eta_V = \frac{x}{\sqrt{4\nu t}}, \quad \eta_H = \frac{x}{\sqrt{4\alpha t}}, \text{ or } \eta_M = \frac{x}{\sqrt{4Dt}}$$

You must write all variables in terms of $\phi, \theta, or\, \psi$ and then use the chain rule, for example:

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial t} = \phi' \frac{x}{\sqrt{4\nu}}\left(-\frac{1}{2}t^{-3/2}\right) \ldots \rightarrow -\frac{\phi'}{2t}\eta = \frac{\nu \phi''}{4\nu t} \rightarrow \phi'' = -2\eta\phi'$$

Now only two conditions are needed to solve the 2nd order ODE. You can use initial conditions or BC's. Also, if we let: $f = \phi'$ then $df/d\eta = -2\eta f$ which leads to: $f = c_0 \exp(-\eta^2) = d\phi/d\eta$. From this we solve for ϕ as:

$$\phi = c_0 \int_0^\eta \exp(-b^2)\, db + C_1 \quad (b \text{ is just a dummy variable})$$

Error Function: is used to evaluate the above integral. The error function is defined as:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \quad \text{where} \quad erf(0) = 0 \text{ and } erf(\infty) = 1$$

Thus we can write: $\phi = c_0 \frac{\sqrt{\pi}}{2} erf(\eta) + c_1$ to which you can apply the conditions discussed above. Also:

$$\phi, \theta, \psi = 1 - erf(\eta) = erfc(\eta)$$ **(complimentary error function** where $\eta = \eta_V, \eta_H, or\, \eta_M$)

Depth of Penetration: defined as value of x at the time t where $\phi, \theta, \psi = 0.01$. It is given by:

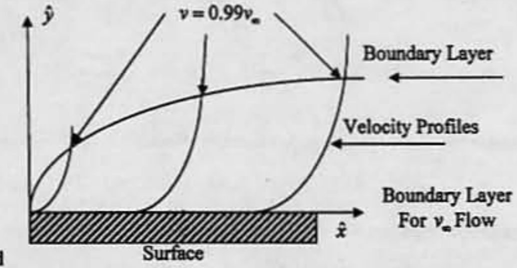
$$\delta = 4\sqrt{\nu t}, \quad \delta_H = 4\sqrt{\alpha t}, \text{ and } \delta_M = 4\sqrt{Dt}$$

The higher the viscosity, the faster linear momentum is transferred to a fluid when in contact with a moving surface, thus causes a high disturbance. Think of honey and water, honey is more viscous and is more disturbed.

Boundary Layer Theory

The thickness of a boundary layer of fluid as a surface moves beneath it, or as the fluid move over a stationary surface is denoted $\delta(x)$. The shape of this profile, which also has analogous profiles for stationary heat ($\delta_T(x)$) or concentration ($\delta_C(x)$) surface boundary layers is shown below for the case where the surface moves to the left or the fluid flows to the right. The boundary layer thickness is the y value at any x where, $\phi, \theta, \psi = 0.99$.

How To Solve Flow Problem For δ: Starting with the Navier-Stokes equation, split it into various component equations for linear momentum and cancel terms that are not necessary. We reason that $v_x >> v_y$ because $\partial x >> \partial y$ (or $L >> \delta$), therefore drop the y linear momentum eq'n that you get from the Navier-Stokes equation. Differentiating the Bernoulli equation outside the boundary gives, $\frac{\partial P}{\partial x} = -\frac{1}{2}\rho\left(2v_\infty \frac{\partial v_\infty}{\partial x}\right) = 0$ which can be subbed



into the x linear momentum eq'n along with $\frac{\partial^2 v_x}{\partial x^2} = 0$ since the velocity gradient in the x direction is small. The x-LM equation becomes: $\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_y}{\partial y}\right) = \nu \left[\frac{\partial^2 v_x}{\partial y^2}\right]$. Similar analogous solutions can be found for T or C_A.

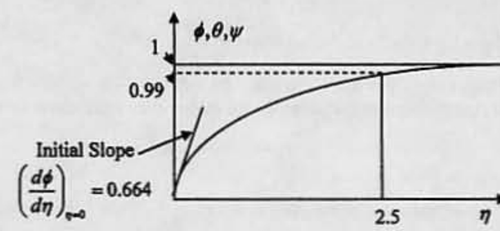
Applying boundary conditions you get the result:

$\eta = \frac{y}{\sqrt{4\nu(x/v_\infty)}}$, (x/v_∞) is the **contact time**

Another important relationship is:

$$\left(\frac{d\psi}{d\eta}\right)_{\eta=0} = \left(\frac{d\theta}{d\eta_T}\right)_{\eta_T=0} = \left(\frac{d\phi}{d\eta_C}\right)_{\eta_C=0} = 0.664$$

which is the initial slope on the diagram to the right:



This shows at $\phi, \theta, \psi = 0.99$ (where the boundary layer is) that $\eta = 2.5$. Letting the height be $y = \delta$, the boundary layer thickness is:

$$\eta = 2.5 = \frac{\delta}{\sqrt{4\nu(x/v_\infty)}} \rightarrow \delta = \frac{5x}{\sqrt{Re_x}}$$

Where $Re_x = \frac{v_\infty x}{\nu}$ is called the **Local Reynolds Number**. This relates to two other dimensionless numbers:

Prandtl Number, Pr: $\frac{\delta}{\delta_T} = Pr^{1/3} = \left(\frac{\nu}{\alpha}\right)^{1/3}$, $\eta_T = \frac{y}{\sqrt{4\alpha(x/v_\infty)}} Pr^{1/3}$

Schmidt Number, Sc: $\frac{\delta}{\delta_C} = Sc^{1/3} = \left(\frac{\nu}{D}\right)^{1/3}$, $\eta_C = \frac{y}{\sqrt{4D(x/v_\infty)}} Sc^{1/3}$

These two numbers relate how fast momentum diffuses compared to mass diffusion and heat respectively.

Friction Coefficient: the shear stress is given by: $\tau_{yx}=\mu\left[\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right]=\mu\frac{\partial v_x}{\partial y}$ since v_y is small. Evaluating at $y=0$ gives: $\tau_0=\left(\tau_{yx}\right)_{y=0}=\mu\left(\frac{\partial v_x}{\partial y}\right)_{y=0}=\mu\frac{dv_x}{d\phi}\left(\frac{d\phi}{d\eta}\right)\frac{\partial\eta}{\partial y}=\mu v_\infty(0.664)\frac{1}{\sqrt{4\nu(x/v_\infty)}}$ which gives the frict. coeff:

$$C_f=\frac{\tau_0}{\frac{1}{2}\rho v_\infty^2}\rightarrow C_f=0.664\left(\frac{\nu}{v_\infty x}\right)^{1/2}\rightarrow C_f=\frac{0.664}{\sqrt{\mathrm{Re}_x}}$$

For an object of length L: $C_{fL}=\frac{1}{L}\int_0^L C_{fx}(x)\,dx=\frac{1.328}{\sqrt{\mathrm{Re}_L}}=2\left(C_{fx}\right)_{x=L}$.

Nusselt Number, Nu: simplifies a similar situation to the linear momentum case for heat as:

$$Nu_x=\frac{h_x x}{k}=0.332(\mathrm{Re}_x)^{1/2}\text{ where }h_x=0.332k\left(\frac{v_\infty}{\nu x}\right)^{1/2}\text{ and }Nu_L=\frac{1}{L}\int_0^L Nu_x(x)\,dx=0.664(\mathrm{Re}_L)^{1/2}$$

Sherwood Number, Sh: simplifies a similar situation to the linear momentum case for diffusion as:

$$Sh_x=\frac{k_x x}{D}=0.332(\mathrm{Re}_x)^{1/2}\text{ where }k_x=0.332D\left(\frac{v_\infty}{\nu x}\right)^{1/2}\text{ and }Sh_L=\frac{1}{L}\int_0^L Sh_x(x)\,dx=0.664(\mathrm{Re}_L)^{1/2}$$

Shear Force: $F=\frac{C_{fL}\frac{1}{2}\rho v_\infty^2}{avg\ \tau_0}\times width\times L$ since Force = Stress x Area.

Stanton Number, St: used to relate some of the dimensionless numbers.

Equating velocity and heat: $\frac{C_f}{2}=St\,\mathrm{Pr}^{2/3}$ where $St=\frac{h/\rho C_P}{v_\infty}$.

Equating velocity and mass transfer: $\frac{C_f}{2}=St\,\mathrm{Pr}^{2/3}=St_{AB}Sc^{2/3}$ where $St_{AB}=\frac{k}{v_\infty}$.

Thus, even if you can't determine the velocity, heat, or mass transfer, you can relate the 3 with these equations.

Other Relationships: using all the information from above to write the shear stress as:

$$\tau_0=C_f\tfrac{1}{2}\rho v_\infty^2\text{ and }\tau_0=\mu\left(\frac{dv_x}{dy}\right)_{y=0}=\mu\left(\frac{dv_x}{d\phi}\right)\left(\frac{d\phi}{d\eta}\right)_{\eta=0}\left(\frac{\partial\eta}{\partial y}\right)$$

Simplifying some of the derivatives as: $\frac{dv_x}{d\phi}=(v_\infty-v_0)=v_\infty$ and $\frac{\partial\eta}{\partial y}=\frac{1}{\sqrt{4\nu(x/v_\infty)}}$, then equating the two gives:

$$\left(\frac{d\phi}{d\eta}\right)_{\eta=0}=\frac{\frac{1}{2}\rho v_\infty C_f}{\mu}\sqrt{4\nu(x/v_\infty)}$$

A similar approach for temperatures gives:

$$\left(\frac{d\phi}{d\eta_T}\right)_{\eta_T=0}=\frac{h}{k}\frac{\sqrt{4\nu(x/v_\infty)}}{\mathrm{Pr}^{1/3}}$$

And lastly for mass transfer:

$$\left(\frac{d\phi}{d\eta_C}\right)_{\eta_C=0}=\frac{k_C}{D}\frac{\sqrt{4\nu(x/v_\infty)}}{Sc^{1/3}}$$

Good Luck on all your exams. We wish you the very best in all your Academics!